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"Joint spectral radius and subdivision schemes"

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The material of Chapter 2 and 3 is joint work with Maria Charina and has been published in the Journal of Computational and Applied Mathematics [Charina and Mejstrik, 2018]. The material of Chapter 4 has been submitted for publication [Mejstrik, 2018a]. Therefore, parts of this thesis can be found verbatim in original research papers.

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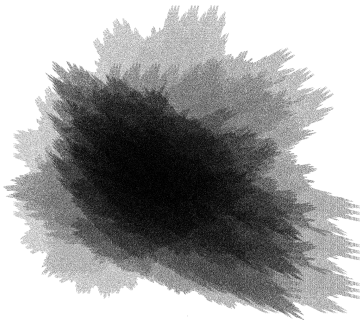

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1 Introduction

Subdivision schemes are fast and robust methods for generating smooth (hyper)-surfaces from a given set of control points, usually in dimensions two or three. We consider the case, when a subdivision operator maps a discrete polygonal mesh of data points, locally isomorphic to $\mathbb{Z}^s$, to a finer mesh of data points in the same space, locally isomorphic to $M^{-1}\mathbb{Z}^s$, where $M \in \mathbb{Z}^{s\times s}$ is a matrix all of whose eigenvalues are greater than one in modulus. The matrix M is referred to as *dilation matrix*. The vertices of the finer mesh are computed by weighted local averages of the coarser mesh. The weights are referred to as the *mask*. A subdivision scheme is an iterated application of subdivision operators on a polygonal mesh, making the mesh finer and finer and eventually, maybe, converging to a smooth limit curve or surface.

We give as an Example the very first subdivision scheme, invented by Rham [1947] and popularized by Chaikin [1974].

Example 1.0.1 (*Corner cutting algorithm*, [Rham, 1947; Chaikin, 1974])**.** Consider the polygonal chain defined by the sequence of points $(c_\alpha)_{\alpha\in\mathbb{Z}}$, $c_\alpha \in \mathbb{R}^2$, with

$$\ldots, \quad c_0 = \begin{bmatrix}0\\2\end{bmatrix}, \quad c_1 = \begin{bmatrix}2\\0\end{bmatrix}, \quad c_2 = \begin{bmatrix}4\\2\end{bmatrix}, \quad c_3 = \begin{bmatrix}2\\1\end{bmatrix}, \quad \ldots$$

plotted in Figure 1.1 (a). For every point $p_\alpha \in \mathbb{R}^2$, $\alpha \in \mathbb{Z}$, we compute two new points $c'_\alpha, c''_\alpha \in \mathbb{R}^2$ as a linear combination of the two neighbouring points with weights $\frac{1}{4}$ and $\frac{3}{4}$, precisely,

$$c'_\alpha = \frac{3}{4}c_\alpha + \frac{1}{4}c_{\alpha-1}, \quad c''_\alpha = \frac{3}{4}c_\alpha + \frac{1}{4}c_{\alpha+1}, \quad \alpha \in \mathbb{Z}.$$

We obtain the sequence $\begin{bmatrix}\ldots & c'_0 & c''_0 & c'_1 & c''_1 & \ldots\end{bmatrix}$ plotted in Figure 1.1 (b), which are refined in the same manner again. In Figure 1.1 we see the generated polygonal chain after one, two and three subdivision steps.

If we define $a = \frac{1}{4}\begin{bmatrix}\mathbf{1} & 3 & 3 & 1\end{bmatrix} \in \ell_0(\mathbb{Z})$, then we can express one subdivision step by

$$c \mapsto Sc = \sum_{\beta\in\mathbb{Z}} a(\,\cdot\, - 2\beta)c(\beta), \quad c \in \ell(\mathbb{Z}, \mathbb{R}^2). \qquad \triangle$$

First subdivision schemes with level independent subdivision weights and uniform underlying mesh, so-called *stationary* schemes like in Example 1.0.1, appeared in the 1960s and are related to the wavelet and frame theory and found applications in signal processing and image

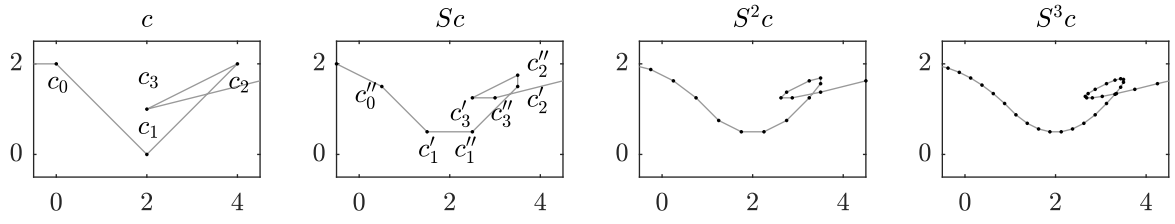


Figure 1.1: Sequence of points as generated in Example 1.0.1.

compression. Because subdivision proofed to be fast, robust and easy to use, the theory of subdivision influenced several applied areas of mathematics and engineering and, in return, has been influenced by applications. Thus, a big variety of subdivision methods were invented. We give a brief, far from exhaustive, overview of various types of subdivision schemes. For a more thorough survey about subdivision schemes, surfaces and methods see e.g. [Sabin, 2005; Peters and Reif, 2008; Cashman, 2012] and the references therein.

- *Non-uniform subdivision* schemes allow for different weights for the computation of each point of the sequence. They are mostly needed for the generation of surfaces with meshes of non-uniform topology, for example surfaces with boundary [Daubechies, Guskov and Sweldens, 1999; Peters and Reif, 2008; Warren and Weimer, 2001].

- *Non-stationary subdivision* (or *level-dependent subdivision*) schemes use different subdivision weights in each step of the iteration. They are used in isogeometric analysis and biological imaging by exploiting their ability to generate and reproduce exponential polynomials [Dyn and Levin, 1995; Cohen and Dyn, 1996; Dyn and Levin, 2002; Charina, Conti, Guglielmi and Protasov, 2016].

- *Multiple subdivision* schemes generalize non-stationary schemes to some extent by allowing in each step of the process a different dilation. Multiple subdivision schemes are building blocks for processing of images with anisotropic directional features and for multigrid methods for solving anisotropic PDEs, see e.g. [Kutyniok and Sauer, 2009; Sauer, 2012; Cotronei, Ghisi, Rossini and Sauer, 2015] and [Charina, Donatelli, Romani and Turati, 2017], respectively.

- *Hermite subdivision* schemes refine not only point positions, but also derivatives or normals [Merrien, 1992; Jüttler and Schwanecke, 2002].

- *Non-linear subdivision* schemes use non-linear rules to generate new vertices, which can help to eliminate artefacts, preserve certain shape properties and allows to generate smooth curves on manifolds [Wallner and Dyn, 2005; Grohs, 2008].

- *Set-valued subdivision* schemes refine general subsets of $\mathbb{R}^n$ [Dyn and Kels, 2011].

One of the most important properties of curves or surfaces generated by subdivision is their smoothness. In the case of level independent or dependent subdivision weights (stationary and non-stationary schemes) this property is well understood and usually characterized either using the matrix or operator approach, see e.g. [Cavaretta, Dahmen and Micchelli, 1991; Dyn and Levin, 2002; Charina, Conti and Sauer, 2005] and [Daubechies and Lagarias, 1992a; Colella and Heil, 1994; Han and Jia, 1998; Chen, Jia and Riemenschneider, 2002; Han, 2002; Cabrelli, Heil and Molter, 2004; Charina and Protasov, 2017], respectively.

The matrix approach studies the spectral properties of finite or compact sets of square matrices derived from the subdivision masks. The essential ingredients are the so-called transition matrices whose entries depend on the subdivision weights and whose structure is inherited from the dilation matrix, i.e. the underlying mesh. The main challenge of adapting the matrix approach to the case of multiple subdivision, and thus level dependent dilation matrices, is in combining the properties of weights and dilations into an appropriate structure of the corresponding transition matrices. Recent advances by Guglielmi and Protasov [2013, 2016] in the exact computation of the joint spectral radius of compact sets of square matrices provide efficient methods for checking both Hölder and Sobolev regularity of subdivision surfaces using such transition matrices.

The operator approach studies the contractivity of the corresponding difference subdivision schemes. The study of the properties of multiple subdivision is at its very beginning. The convergence analysis of multiple subdivision in terms of the restricted spectral radius is given in [Sauer, 2012].

1.1 Main results and overview

The aim of this thesis is to adapt the matrix approach to the setting of multiple subdivision and to develop algorithms for checking the regularity of multiple subdivision schemes.

Chapter 2 studies compact subsets of $\mathbb{R}^s$, which are the fixed points of finite sets of contraction mappings. These sets, the so-called *attractors* can be seen as the sets of all (multivariate) numbers in (multiple) positional number systems. The support of functions generated by subdivision schemes can be described using such sets, and, subsequently, their structure is crucial for the construction of the transition matrices in Chapter 3.

Most of the results in this chapter generalize well known statements about attractors from the stationary case to the multiple case, see Lemmata 2.1.14, 2.1.15, 2.1.16 and Corollaries 2.1.19, 2.1.21.

However, there are some differences between those two cases, pointed out in Examples 2.2.3 and 2.2.4. Furthermore, the Lebesgue measure of the boundary of attractors in the multiple case remains an open problem, see Example 2.2.5 and Proposition 2.2.6.

For numerical applications we implement the functions (*i*) `checktile` which implements an algorithm by Gröchenig and Haas [1994], testing whether an attractor is a tile or not, (*ii*) `tile` plotting multiple attractors and (*iii*) `constructdigit` which implements an algorithm by Cotronei, Ghisi, Rossini and Sauer [2015] constructing digit sets.

Chapter 3 develops the theory for the matrix approach for multiple subdivision.

In Section 3.1, we adapt the basic concepts of stationary subdivision to the multiple case.

In Section 3.2 and 3.3 we give the first explicit construction of transition matrices for the multiple case. Even in the stationary case it was not always possible before to construct transition matrices for certain dilation matrices. The invariant Omega algorithm 3.2.9 settles this problem. Our construction yields the smallest possible transition matrices whose index set contains $0 \in \mathbb{Z}^s$. Conjecture 3.3.4 stipulates the smallest possible size of transition matrices without the restriction that 0 is in the index set.

In Section 3.4 we investigate the space of polynomial sequences, which has different properties in the multiple subdivision setting for general dilations matrices, compared to the stationary setting, see Example 3.4.19.

The main result in this chapter, Theorem 3.4.17, relates the convergence analysis of multiple subdivision and the joint spectral radius techniques and allows us to use the efficient methods by Guglielmi and Protasov [2013], studied in Chapter 4.

For numerical applications we implement the functions (*i*) `blf` plotting the basic limit functions of multiple subdivision schemes, (*ii*) `constructOmega` constructing the smallest invariant space invariant under transition operators and (*iii*) `constructV`, `constructVt` constructing sequence spaces orthogonal to polynomial sequences.

Chapter 4 discusses the explicit computation of the joint spectral radius of a finite set of square matrices by using and improving the invariant polytope algorithm by Guglielmi and Protasov [2013, 2016]. The invariant polytope algorithm is one of the only algorithms which can compute the exact value of the joint spectral radius.

In Section 4.1 and 4.2, we recall properties about the joint spectral radius and explain the invariant polytope algorithm.

In Section 4.3, we develop the modified Gripenberg algorithm 4.3.5 which gives very good lower bounds for the joint spectral radius in a very short time.

In Section 4.4, we propose modifications for the invariant polytope algorithm making it faster, more robust and easier to use. Furthermore, we give a counter example to a conjecture regarding the balancing of multiple cyclic trees in Section 4.4.4, propose a better balancing procedure for multiple cyclic trees in Section 4.4.5 and prove that the intermediate bounds for the JSR

computed by the modified invariant polytope algorithm are correct in all cases. The original invariant polytope algorithm sometimes returns wrong bounds.

Using the modified invariant polytope algorithm, we compute the regularity of Daubechies wavelets of high order in Section 4.5.4, refine an observation about the regularity of Daubechies wavelets from [Guglielmi and Protasov, 2015] in Remark 4.5.11 and compute the exact capacity of codes with forbidden differences for many new examples in Section 4.5.3.

For numerical applications we implement the functions (*i*) `tjsr` computing the joint spectral radius using the modified invariant polytope algorithm 4.4.1, (*ii*) `findsmp` which computes lower bounds for the joint spectral radius using the modified Gripenberg algorithm 4.3.5 and (*iii*) `codecapacity` which implements an algorithm by Moision, Orlitsky and Siegel [2001], constructing matrices whose joint spectral radius is related to the capacity of codes which avoid certain forbidden differences.

In the appendix a comprehensive **Glossary** and list of **Symbols** can be found. The source code of simple implementations of the invariant Omega algorithm 3.2.9 and the modified Gripenberg algorithm 4.3.5 are listed in Sections 3.6 and 4.6, respectively. The source code of all mentioned programs can be found at **tommsch.com/science** or comes together with this thesis.

1.2 Definitions and Notation

The following notation is used throughout the thesis.

• We denote by $\mathbb{C}$ the *complex numbers*, by $\mathbb{R}$ the *real numbers*, by $\mathbb{R}_+$ the *positive real numbers including zero*, by $\mathbb{Q}$ the *rational numbers*, by $\mathbb{Z}$ the *integers*, by $\mathbb{N}$ the *positive integers without zero* and by $\mathbb{N}_0$ the *positive integers including zero*. By $\emptyset$ we denote the empty set. For $z \in \mathbb{C}$, we denote its real part by $\Re(z)$ and its imaginary part by $\Im(z)$.

• Given sets A, B, we write $A \subsetneqq B$ if A is a *strict subset* of B and $A \subseteq B$ if either $A = B$ or $A \subsetneqq B$. We write $A \not\subseteq B$ if A is not a subset of B. The *set difference* of A and B is denoted by $A \setminus B = \{a \in A : a \notin B\}$. For a finite set A, we denote the *number of elements* in A by $\#A$.

Given $A, B \subseteq \mathbb{R}^s$. We denote the *Lebesgue measure* of A by $\lambda(A)$, if it exists. We say that A and B are *essentially equal*, denoted by $A \simeq B$, if they only differ by a set of Lebesgue-measure zero. We define $A \pm B = \{a \pm b : a \in A,\ b \in B\}$ and $AB = \{ab : a \in A,\ b \in B\}$. For $M \in \mathbb{R}^{s\times s}$ we define $MA = \{Ma : a \in A\}$. For a finite set of matrices $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$, $J \in \mathbb{N}$, we denote with $\mathcal{A}^n$, $n \in \mathbb{N}$, all matrix products of length n with matrices from $\mathcal{A}$.

Let X be a topological space and $A \subseteq X$. We denote by A° the *interior* of A, by $\mathrm{cl}(A)$ the *closure* of A and by ∂A the *boundary* of A.

• Let $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$. An element of $\mathbb{N}_0^s$ is called *multi-index* and written as a row-vector within parentheses. Given $\mu, \nu \in \mathbb{N}_0^s$ multi-indices, we define the *length* $|\mu| = \mu_1 + \dots + \mu_s$, the *sum* and the *difference* $\mu \pm \nu = (\mu_1 \pm \nu_1, \dots, \mu_s \pm \nu_s)$ whenever the right hand side is defined, a *partial ordering* $\nu \le \mu \Leftrightarrow \nu_l \le \mu_l$ for all $l \in \{1, \dots, s\}$ and the *factorial* $\mu! = \mu_1! \cdots \mu_s!$. If $\nu \le \mu$, then we define the *binomial coefficient* by

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu-\nu)!} = \prod_{l=1}^{s} \binom{\mu_l}{\nu_l} = \binom{\mu_1}{\nu_1}\binom{\mu_2}{\nu_2}\cdots\binom{\mu_s}{\nu_s}. \tag{1.2.1}$$

For $z \in \mathbb{C}^s$, $M \in \mathbb{Z}^{s\times s}$, $\alpha \in \mathbb{Z}^s$, we define the *power* $z^\alpha = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ and the *matrix power* $z^M = (z^{M_1}, \dots, z^{M_s})$, where M_l denotes *the* l^{th} *column* of M.

• Let X, Y be sets, X countable. We denote by Y^X or $\ell(X, Y)$ the set of *sequences* of elements in Y indexed by elements in X. If $Y = \mathbb{R}$, then we also write $\ell(X)$. An element of Y^X is denoted by $(y_x)_{x\in X}$, $(y_x)_x$ or $\boldsymbol{y}$.

For a sequence $a \in \ell(X, Y)$ we denote the value at index $x \in X$ with $a(x)$ or a_x. The *support* of a sequence $a \in \ell(X)$ is defined by $\operatorname{supp} a = \{x \in X : a(x) \neq 0\}$. With $\ell_0(X)$ we denote finitely supported sequences. With $\ell_p(\mathbb{Z}^s)$, $1 \leq p < \infty$, we denote sequences with finite p-Norm defined by $||c||_p^p = ||c||_{\ell_p}^p = \sum_{\alpha\in\mathbb{Z}^s} ||c(\alpha)||_p^p$, where $||c(\alpha)||_p$ is the p-vector norm of $c(\alpha)$ defined by $||c(\alpha)||_p^p = \sum_{l=1}^s |c(\alpha)_l|^p$. With $\ell_\infty(\mathbb{Z}^s)$ we denote bounded sequences with finite supremum norm defined by $||c||_\infty = ||c||_{\ell_\infty} = \sup_{\alpha\in\mathbb{Z}^s} |c(\alpha)|$.

The *bold number* in a sequence or matrix denotes the zeroth entry. We denote with $\delta \in \ell(\mathbb{Z}^s)$ the *Kronecker delta*, $\delta = \big[\mathbf{1}\big]$, i.e. $\delta(\alpha) = 1$ for $\alpha = 0$ and zero otherwise.

We identify finite sequences with vectors, matrices and tensors, and thus we settle for the choice to write sequences in $\ell(\mathbb{Z})$ as column-vectors. Given $A \in \mathbb{R}^{s\times s}$, we denote by A^T the *transpose* of A, by A^* the *conjugate transpose* of A, by $\rho(A)$ the *spectral radius* of A and by $\det(A)$ the *determinant* of A, We denote with I the *identity matrix* and with e_l the l^{th} *standard unit vector* of $\mathbb{R}^s$.

Given countable sets I, J and $m_{i,j} \in \mathbb{R}$ for $i \in I$, $j \in J$. We say $(m_{i,j})_{i\in I, j\in J} \in \mathbb{R}^{I\times J}$ is a *generalized* $I \times J$ *matrix* [Cabrelli, Heil and Molter, 2004, Section 2.1]. Generalized matrices may always be realized as ordinary matrices by choosing a specific ordering for the sets I and J.

All sequences and matrices are written within square-brackets $[\,\cdot\,]$. Closed and open intervals are denoted by $[\quad]$ and $(\quad)$, respectively.

• For $\alpha \in (0, 1]$ we denote by $C^\alpha(\mathbb{R}^s)$ the Hölder continuous functions with exponent α, i.e. $f \in C^\alpha$ if there exists $C > 0$ such that $|f(x) - f(y)| \leq C\,||x - y||^\alpha$ for all $x, y \in \mathbb{R}^s$.

• We mostly use the letters α, β, γ as indices and the letters f, g, ϕ, ψ for functions. We usually denote by c a *sequence*, by a a finitely supported *mask sequence*, by M a *dilation matrix*, by s the dimension, by l the running index of the dimension, and by i we denote either an index or the imaginary unit $i^2 = -1$. *Index sequences* are usually denoted by $\boldsymbol{j}$, $\boldsymbol{j} \in \{1, \ldots, J\}^{\mathbb{N}}$, and their elements are denoted by j_1, j_2, etc., i.e. without using bold font.

• The *floor function*, taking $x \in \mathbb{R}$ and returning the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$. The *linear span* (or *linear hull*) of a set V is denoted by $\operatorname{span} V$. The *inner product* of $x, y \in \mathbb{R}^s$ is denoted by (x, y).

We abbreviate JSR for *joint spectral radius*, RSR for *restricted spectral radius* and *LP* for linear programming. The shorthand notation $j = 1, \ldots, J$ is used to denote $j \in \{1, \ldots, J\}$. For $a, b \in \mathbb{R}$ we write $a \simeq b$ if the values of a and b are approximately equal. The symbol $\square$ closes proofs, the symbol $\triangle$ closes examples.

Having established ourselves on this sound basis, it is our duty to see what inferences can be drawn [Doyle, 1893].

2 Attractors and tiles

In this chapter we study the basic ingredient for the matrix approach to subdivision schemes; compact subsets of $\mathbb{R}^s$, $s \in \mathbb{N}$, which are the fixed points of a set of contractive mappings. Given a bounded sequence of jointly expanding dilation matrices M_j and a bounded sequence of finite subsets $Q_j \subseteq \mathbb{Z}^s$, we investigate the corresponding *attractor*, i.e. the set of numbers which can be represented in a positional number system with matrix-bases M_j and digits $q \in Q_j$. A common positional number system is the decimal number system with digits 0 to 9 and base 10, whose corresponding attractor is the interval $[0, 1]$. A common multiple positional number system is used to denote the time with bases 24, 60, 60 and 10, denoting hours, minutes, seconds, and fractions of seconds, respectively.

The name *attractor* is borrowed from the stationary case where only one dilation matrix $M \in \mathbb{Z}^{s\times s}$ and one finite subset $Q \subseteq \mathbb{Z}^s$ is considered [Barnsley, 1988]. In this setting, the corresponding attractor is the unique compact set $K_Q \subseteq \mathbb{R}^s$ which satisfies the fixed point equation

$$K_Q = M^{-1}(K_Q + Q), \quad K_Q \subseteq \mathbb{R}^s.$$

2.1 Attractors

The joint spectral radius (JSR) of a finite set of matrices, is a quantity which describes the worst case growth rate of the norms of products of matrices from this set. It is a generalization of the standard spectral radius of one matrix.

The notion of the joint spectral radius plays a central role in this work. In the context of multiple attractors, the topic of Chapter 2, the concept of the joint spectral radius is used to describe the joint expanding properties of several dilation matrices. In the context of multiple subdivision, the topic of this Chapter, the concept of the joint spectral radius is used to describe the joint contracting properties of operators corresponding to the considered subdivision operators. Finally, in Chapter 4, we discuss the joint spectral radius itself and how to compute it.

Definition 2.1.1 ([Rota and Strang, 1960])**.** Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$ be a finite set of square matrices. The *joint spectral radius* (JSR) of $\mathcal{A}$ is defined by

$$\mathrm{JSR}(\mathcal{A}) = \lim_{n\to\infty} \max_{A_j\in\mathcal{A}} ||A_{j_n} \cdots A_{j_1}||^{1/n}.$$

The limit in Definition 2.1.1 exists, is independent of the matrix norm and can be expressed as the infimum over all possible norms [Rota and Strang, 1960, Proposition 1]. Precisely, if $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$ is a finite set of square matrices, then

$$\mathrm{JSR}(\mathcal{A}) = \inf_{||\cdot|| \; A_j\in\mathcal{A}} \max ||A_j||. \tag{2.1.1}$$

Since we will use this statement later on, we provide its proof in Chapter 4, Lemma 4.1.1.

We use the JSR to describe the joint expanding properties of several matrices. A single matrix is expanding, if all of its eigenvalues are greater than one in modulus or, equivalently, all eigenvalues of the inverse matrix are less than one in modulus, and thus the spectral radius of

the inverse is less than one. From this observation, the definition of jointly expanding matrices is straightforward.

Definition 2.1.2. A finite set of invertible matrices $\{M_j \in \mathbb{Z}^{s\times s} : j = 1, \dots, J\}$ is *jointly expanding* if

$$\mathrm{JSR}(\Big\{M_j^{-1} : j = 1, \dots, J\Big\}) < 1.$$

A matrix $M \in \mathbb{Z}^{s\times s}$ is a *dilation matrix* if $\rho(M^{-1}) < 1$.

We want to study compact sets which are the fixed point of a finite set of affine contractive mappings. The natural space to work with is the Hausdorff-space $\mathcal{H}(\mathbb{R}^s)$.

Definition 2.1.3 ([Hausdorff, 1914, page 294])**.** Let $K \subseteq \mathbb{R}^s$. We define $\mathcal{H}(K)$ as the space of all non-empty, compact subsets of K with the *Hausdorff metric* h defined by

$$\mathrm{h}(B, C) = \max\left\{\sup_{b\in B}\inf_{c\in C}||b-c||_2\,,\,\sup_{c\in C}\inf_{b\in B}||b-c||_2\right\},\quad B, C \in \mathcal{H}(K). \tag{2.1.2}$$

The topology induced by the Hausdorff metric is independent of the norm used in (2.1.2) and if $K \subseteq \mathbb{R}^s$ is compact, then the space $\mathcal{H}(K)$ is a complete metric space [Hausdorff, 1914, page 294]. The induced topology is rather coarse, as the next example shows.

Example 2.1.4. Let $n \in \mathbb{N}$. If $A_n = \frac{1}{n}\mathbb{Z} \cap [0, 1]$, then $A_n \to [0, 1]$ as $n \to \infty$ in the Hausdorff metric h. Indeed, for $n \in \mathbb{N}$ it follows that

$$\begin{aligned}\sup_{a_n\in A_n}\inf_{a\in[0,1]}||a-a_n||_2 &= 0,\\ \sup_{a\in[0,1]}\inf_{a_n\in A_n}||a-a_n||_2 &= \frac{1}{2n},\end{aligned}$$

and therefore $\mathrm{h}(A_n, [0, 1]) = \frac{1}{2n}$. Note that, $\#A_n < \infty$, $n \in \mathbb{N}$, whereas $[0, 1]$ is uncountable. △

Definition 2.1.5. Let $\{M_j \in \mathbb{Z}^{s\times s} : j = 1, \dots, J\}$ be jointly expanding and $Q_j \subseteq \mathbb{Z}^s$, $j \in \{1, \dots, J\}$, be finite. We define the *(multiple) attractor* associated to Q_j, $j \in \{1, \dots, J\}$ and $\boldsymbol{j} \in \{1, \dots, J\}^{\mathbb{N}}$ by

$$K_{Q,\boldsymbol{j}} = \mathrm{cl}_{\mathcal{H}}\left(M_{j_1}^{-1}Q_{j_1} + M_{j_1}^{-1}M_{j_2}^{-1}Q_{j_2} + \cdots\right) = \mathrm{cl}_{\mathcal{H}}\left(\sum_{m=1}^{\infty}\Big(\prod_{n=1}^{m}M_{j_n}^{-1}\Big)Q_{j_m}\right). \tag{2.1.3}$$

For $J = 1$, we write $K_{Q,\boldsymbol{j}} = K_Q$ and call it *stationary attractor*.

For $x = M_{j_1}^{-1}q_1 + M_{j_1}^{-1}M_{j_2}^{-1}q_2 + \cdots \in K_{Q,\boldsymbol{j}}$, $q_n \in Q_{j_n}$, we also write

$$x = .q_1q_2\dots\ . \tag{2.1.4}$$

and call it the *expansion* of x.

In view of (2.1.4), attractors can be seen as the set of points in a positional number system, with no digits left to the radix point. We present examples for the multiple case later.

Example 2.1.6. Let $s = 1$, $M = 3$ and $Q = \{0, 2\}$. The corresponding attractor K_Q is plotted in Figure 2.1. It is the Cantor set, discovered by Smith [1874] and discussed by Cantor in 1883, created by iteratively deleting the open middle third from a set of line segments. This can be seen using (2.1.4). Indeed, the set K_Q consists of exactly those numbers in base 3 whose expansions do not contain the digit 1; which is exactly the Cantor set. △

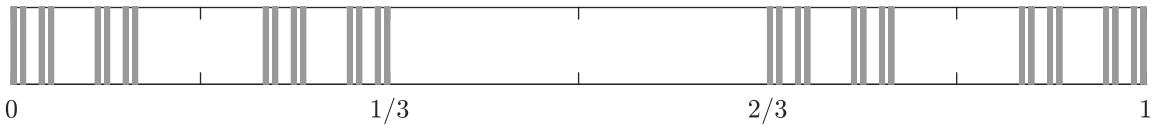


Figure 2.1: Attractor from Example 2.1.6.

Remark 2.1.7. The function `tile` [Mejstrik, 2018b] plots multiple attractors and is used to plot all pictures of attractors in this thesis.

We are mainly interested in attractors associated to two types of sets:

(*i*) supports of mask sequences, defined in Chapter 3, whose attractors are the supports of the functions generated by the corresponding convergent subdivision schemes.

(*ii*) digit sets corresponding to dilation matrices, whose attractors are used to build a cover for the sets from (*i*), see Chapter 3.3.

We define digit sets next.

Definition 2.1.8. A *digit set* $D \subseteq \mathbb{Z}^s$ corresponding to a dilation matrix M is a complete set of representatives of the quotient group $\mathbb{Z}^s/M\mathbb{Z}^s = \{\alpha + M\mathbb{Z}^s : \alpha \in \mathbb{Z}^s\}$, i.e. $D \simeq \mathbb{Z}^s/M\mathbb{Z}^s$. The elements of a digit set are called *digits*.

Any digit set has as many elements as the modulus of the determinant of its corresponding dilation matrix. A proof for this well known statement can be found in [Gröchenig and Madych, 1992, Lemma 2].

Lemma 2.1.9. *If M is a dilation matrix and $D \simeq \mathbb{Z}^s/M\mathbb{Z}^s$, then $\#D = |\det M|$.*

Example 2.1.10. (*i*) For the matrix $M = 10$ one possible digit set is $D = \{0, \dots, 9\}$. The corresponding attractor is the interval $[0, 1]$.

(*ii*) Let $s = 2$, $M_{\mathbb{DD}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $D_{\mathbb{DD}} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. The corresponding attractor $K_{\mathbb{DD}} = K_{D_{\mathbb{DD}}}$ is usually called *double dragon* and can be seen in Figure 2.2. △

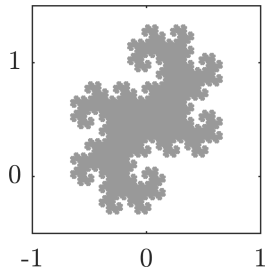


Figure 2.2: The so-called *double dragon* $K_{\mathbb{DD}}$ from Example 2.1.10 (*ii*). A really beautiful plot of this attractor (and others) can be found in [Gilbert, 1981]. Nice plots of three headed and tame dragons can be found in [Bandt, 1991].

In Lemma 2.1.11 we present two standard choices for digit sets and their straightforward construction [Cotronei, Ghisi, Rossini and Sauer, 2015].

Lemma 2.1.11. (*i*) *Let $M \in \mathbb{Z}^{s\times s}$ be a dilation matrix. The image of $[0,1)^s$ under M intersected with the integers is a digit set for M, i.e.*

$$M[0,1)^s \cap \mathbb{Z}^s \simeq \mathbb{Z}^s/M\mathbb{Z}^s. \tag{2.1.5}$$

It can be computed by

$$M[0,1)^s \cap \mathbb{Z}^s = \left\{\alpha - M\left\lfloor M^{-1}\alpha\right\rfloor : \alpha \in \mathbb{Z}^s\right\},$$

where the floor function $\lfloor\cdot\rfloor$ is defined component-wise.

(*ii*) *Let $M_1, M_2 \in \mathbb{Z}^{s\times s}$ be jointly expanding. If $D_1 \subseteq \mathbb{Z}^s$ is a digit set for the dilation matrix M_1 and $D_2 \subseteq \mathbb{Z}^s$ is a digit set for the dilation matrix M_2, then $M_2D_1 + D_2$ is a digit set for the dilation matrix M_2M_1.*

Proof. (*i*) We start by showing (2.1.5). Note that, since $M([0,1)^s + \mathbb{Z}^s) = \mathbb{R}^s \supseteq \mathbb{Z}^s$, it follows that for all $\alpha \in \mathbb{Z}^s$, there exists $\beta \in M\mathbb{Z}^s$ such that $\alpha + \beta \in M[0,1)^s$. It remains to show that all elements of $M[0,1)^s \cap \mathbb{Z}^s$ are distinct representatives of $\mathbb{Z}^s/M\mathbb{Z}^s$. Let $\alpha, \beta \in M\mathbb{Z}^s$ such that $\alpha - \beta \in M\mathbb{Z}^s$. Thus, $\alpha - \beta \in \big((M[0,1)^s \cap \mathbb{Z}^s) - (M[0,1)^s \cap \mathbb{Z}^s)\big) \cap M\mathbb{Z}^s$ which implies that $\alpha - \beta \in M(-1,1)^s \cap \mathbb{Z}^s \cap M\mathbb{Z}^s = M(-1,1)^s \cap M\mathbb{Z}^s = 0$. Therefore, (2.1.5) is proven.

Equation (2.1.6) is straightforward to show after multiplying it with M^{-1}, which gives

$$[0,1)^s \cap M^{-1}\mathbb{Z}^s = \left\{\gamma - \lfloor\gamma\rfloor : \gamma \in M^{-1}\mathbb{Z}^s\right\}. \tag{2.1.6}$$

Since $\lfloor\gamma\rfloor = 0$ for $\gamma \in [0,1)^s$ it follows that $[0,1)^s \cap M^{-1}\mathbb{Z}^s \subseteq \left\{\gamma - \lfloor\gamma\rfloor : M^{-1}\gamma \in \mathbb{Z}^s\right\}$. Since the floor function maps (component wise) a number $x \in \mathbb{R}$ to the greatest integer less than or equal to x, i.e. $\lfloor x\rfloor \leq x$ and $x - \lfloor x\rfloor \in [0,1)$, the other inclusion follows.

(*ii*) First note that all elements $d \in M_2D_1$ are representatives of the class $0 \in \mathbb{Z}^s/M_2\mathbb{Z}^s$. Thus, by Lemma 2.1.9, $M_2D_1 + D_2$ has $|\det M_2 \cdot \det M_1|$ elements.

We show next that all elements of $M_2D_1 + D_2$ are different representatives of elements of the quotient group $\mathbb{Z}^s/M_2M_1\mathbb{Z}^s$. Let $a_1, b_1 \in D_1$, $a_2, b_2 \in D_2$ and define $a = M_2a_1 + a_2$, $b = M_2b_1 + b_2$. Assume a and b represent the same class in $\mathbb{Z}^s/M_2M_1\mathbb{Z}^s$. Thus, there exists $\alpha \in \mathbb{Z}^s$ such that $M_2a_1 + a_2 - M_2b_1 - b_2 = M_2M_1\alpha$ or, equivalently, $M_2(a_1 - b_1 - M_1\alpha) = b_2 - a_2$. Therefore, $b_2 - a_2 \in M_2\mathbb{Z}^s$ from which it follows that $a_2 = b_2$, and subsequently, with the same argument, $a_1 = b_1$. Finally, the claim follows by Lemma 2.1.9. □

Note that these two constructions can yield different sets.

Example 2.1.12. Let $M_1 = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, $M_2 = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ and define $M_{21} = M_2M_1$. The digit sets D_1, D_2 and $D_{21}^{(a)}$ corresponding to M_1, M_2 and M_2M_1, respectively, constructed using Lemma 2.1.11 (*i*), are $D_1 = M_1[0,1)^s \cap \mathbb{Z}^s = \{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\}$, $D_2 = M_2[0,1)^s \cap \mathbb{Z}^s = \{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\}$ and $D_{21}^{(a)} = M_2M_1[0,1)^s \cap \mathbb{Z}^s = \left\{\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right\}$. However, the digit set $D_{21}^{(b)}$, constructed using D_1, D_2 and Lemma 2.1.11 (*ii*), is $D_{21}^{(b)} = M_2D_1 + D_2 = \left\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\-1\end{smallmatrix}\right]\right\}$. △

Remark 2.1.13. The function `constructdigit` [Mejstrik, 2018b] implements an algorithm cited in [Cotronei, Ghisi, Rossini and Sauer, 2015, Lemma 1] which constructs, for a given dilation matrix M, the digit set $M[0,1)^s \cap \mathbb{Z}^s$. Note that there is a typo in that paper; the authors use integer-truncation instead of the floor-function.

The following properties of the attractors are reminiscent of the stationary and non-stationary settings. Lemma 2.1.14 shows that multiple attractors are uniformly bounded, in particular compact.

Lemma 2.1.14. *If $\{M_j \in \mathbb{Z}^{s\times s} : j = 1,\dots,J\}$ is jointly expanding and $Q_j \subseteq \mathbb{Z}^s$, $j \in \{1,\dots,J\}$, finite, then there exists $C > 0$ such that for all $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$,*

$$||x||_2 \le C \quad \text{for all} \quad x \in K_{Q,\boldsymbol{j}}$$

In particular, $K_{Q,\boldsymbol{j}}$ is compact.

Proof. The boundedness of $K_{Q,\boldsymbol{j}}$ follows, by (2.1.1), due to the existence of a matrix norm $||\cdot||$ and constant $C_M > 0$ such that $C_M = \max_{j\in\{1,\dots,J\}} \left|\left|M_j^{-1}\right|\right| < 1$, and the fact that the sets Q_j, $j \in \{1,\dots,J\}$, are finite, i.e. bounded by a constant $0 < C_Q < \infty$. Indeed, for every $x = .q_1q_2\ldots \in K_{Q,\boldsymbol{j}}$, $q_n \in Q_{j_n}$,

$$||x|| = \left|\left|M_{j_1}^{-1}q_1 + M_{j_1}^{-1}M_{j_2}^{-1}q_2 + \cdots\right|\right| \le C_Q \sum_{n=1}^{\infty} C_M^n = \frac{C_Q C_M}{1-C_M}.$$

Note that C_M and C_Q do not depend on $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$. The attractor $K_{Q,\boldsymbol{j}}$ is closed by definition, thus compact. □

Proposition 2.1.15, a direct generalization of [Gröchenig and Madych, 1992, Lemma 3] to the multiple case, gives an explicit construction for the attractors $K_{Q,\boldsymbol{j}}$ defined in (2.1.3). It will become helpful in the forthcoming proofs.

Proposition 2.1.15. *Let $\{M_j \in \mathbb{Z}^{s\times s} : j = 1,\dots,J\}$ be jointly expanding matrices, $Q_j \subseteq \mathbb{Z}^s$, $j \in \{1,\dots,J\}$, be finite and $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$. If $\tilde{K}_{0,n} \subseteq \mathbb{R}^s$, $n \in \mathbb{N}$, is a sequence of compact, non-empty and uniformly bounded sets, i.e. there exists $C > 0$ such that*

$$||\tilde{x}_0||_2 \le C, \quad \tilde{x}_0 \in \tilde{K}_{0,n},\ n \in \mathbb{N},$$

then the sequence of compact sets

$$\begin{aligned}\tilde{K}_n &= M_{j_1}^{-1}Q_{j_1} + \cdots + M_{j_1}^{-1}\cdots M_{j_n}^{-1}Q_{j_n} + M_{j_1}^{-1}\cdots M_{j_n}^{-1}\tilde{K}_{0,n} \\ &= \sum_{m=1}^{n}\left(\prod_{k=1}^{m} M_{j_k}^{-1}\right)Q_{j_m} + \left(\prod_{k=1}^{n} M_{j_k}^{-1}\right)\tilde{K}_{0,n}, \quad n \in \mathbb{N}.\end{aligned} \tag{2.1.7}$$

converges to $K_{Q,\boldsymbol{j}}$ with respect to the Hausdorff-metric.

Proof. As in the proof of Lemma 2.1.14, by (2.1.1), there exists a norm $||\cdot||$ and constants $C_M, C_Q, C_0 > 0$ such that $C_M = \max_{j\in\{1,\dots,J\}} \left|\left|M_j^{-1}\right|\right| < 1$, $||x|| < C_Q$ for all $x \in K_{Q,\boldsymbol{j}}$ and $||\tilde{x}_0|| < C_0$ for all $\tilde{x}_0 \in \tilde{K}_{0,n}$, $n \in \mathbb{N}$. Let $\tilde{x}_n \in \tilde{K}_n$. By (2.1.7), it follows that there exist $q_m \in Q_{j_m}$, $m \in \{1,\dots,n\}$, such that $\tilde{x}_n = .q_1\ldots q_n + y_n$, where $y_n \in M_{j_1}^{-1}\cdots M_{j_n}^{-1}\tilde{K}_{0,n}$, $||y_n|| < C_M^{-n}C_0$. Therefore, defining $x = .q_1\ldots q_nq_{n+1}q_{n+2}\ldots \in K_{Q,\boldsymbol{j}}$, with arbitrary $q_m \in Q_{j_m}$, $m \ge n+1$, we get

$$\begin{aligned}||\tilde{x}_n - x|| &= ||.q_1\ldots q_n + y - .q_1q_2\ldots|| = ||-.0\ldots 0q_{n+1}q_{n+2}\ldots + y_n|| \\ &\le ||.0\ldots 0q_{n+1}q_{n+2}\ldots|| + ||y_n|| \le C_M^{-n}C_Q + C_M^{-n}C_0.\end{aligned}$$

Thus, $\sup_{\tilde{x}_n\in\tilde{K}_n}\inf_{x\in K}||x-\tilde{x}_n|| \to 0$ as $n\to\infty$.

For the other estimate needed for (2.1.2), choose $x = .q_1q_2\ldots \in K_{Q,\boldsymbol{j}}$, define for $n \in \mathbb{N}$ the sequences $\tilde{x}_n = .q_1\ldots q_n + M_{j_1}^{-1}\cdots M_{j_n}^{-1}\tilde{x}_{0,n}$, with some $\tilde{x}_{0,n} \in \tilde{K}_{0,n}$, and show that $\tilde{x}_n \to x$ as $n\to\infty$. □

Corollary 2.1.16 of Proposition 2.1.15 is a direct generalization of [Gröchenig and Madych, 1992, Theorem 2.3]. It shows that integer translates of multiple attractors cover $\mathbb{R}^s$. We will elaborate on this property in Section 2.2.

Corollary 2.1.16. *If $\{M_j \in \mathbb{Z}^{s\times s} : j = 1, \dots, J\}$ is jointly expanding with corresponding digit sets $D_j \simeq \mathbb{Z}^s/M_j\mathbb{Z}^s$ and $\boldsymbol{j} \in \{1, \dots, J\}^{\mathbb{N}}$, then*

(i) $K_{D,\boldsymbol{j}} + \mathbb{Z}^s = \mathbb{R}^s$,

(ii) $\lambda\left(K^{\circ}_{D,\boldsymbol{j}}\right) \neq 0$ *and*

(iii) $\lambda\left(K_{D,\boldsymbol{j}}\right) \geq 1$.

Proof. *(i)* Let $\tilde{K}_0 = [0,1]^s$ and define for $n \in \mathbb{N}$, $\tilde{K}_n = M_{j_1}^{-1}D_{j_1} + \cdots + M_{j_1}^{-1}\cdots M_{j_n}^{-1}D_{j_n} + M_{j_1}^{-1}\cdots M_{j_n}^{-1}\tilde{K}_0$ and $D_{j^{[1,n]}} = M_{j_n}\cdots M_{j_2}D_{j_1} + M_{j_n}\cdots M_{j_3}D_{j_2} + \cdots + D_{j_n}$. First observe that

$$\tilde{K}_n + \mathbb{Z}^s = \mathbb{R}^s, \quad n \in \mathbb{N}, \tag{2.1.8}$$

since each D_j is a digit set for the dilation matrix M_j, $j \in \{1, \dots, J\}$. Indeed, $\tilde{K}_n + \mathbb{Z}^s = \mathbb{R}^s$ if and only if $\tilde{K}_0 + D_{j^{[1,n]}} + M_{j_n}\cdots M_{j_1}\mathbb{Z}^s = \mathbb{R}^s$ and $D_{j^{[1,n]}}$ is a digit set for the dilation matrix $M_{j_n}\cdots M_{j_1}$ by Lemma 2.1.11 *(ii)*. Thus, (2.1.8) holds.

Let $x \in \mathbb{R}^s$. We now show that there exists $\alpha \in \mathbb{Z}^s$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $x+\alpha \in K_{j_{n_k}}$ for all $k \in \mathbb{N}$. By the preliminary observation 2.1.8, there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of lattice points $\alpha_n \in \mathbb{Z}^s$ such that $x+\alpha_n \in \tilde{K}_n$ for all $n \in \mathbb{N}$. Since the sets $\tilde{K}_n$ are bounded, the sequence $(\alpha_n)_n$ is bounded. Therefore, there exists a constant subsequence $(\alpha_{n_k})_k = \alpha$, which implies $x + \alpha \in \tilde{K}_{n_k}$ for all $k \in \mathbb{N}$. Since, by Proposition 2.1.15, the sets $\tilde{K}_n$ converge in the Hausdorff-metric (2.1.2) to $K_{D,\boldsymbol{j}}$, it follows that $x \in K_{D,\boldsymbol{j}}$. Otherwise, $x \in (\mathbb{R}^s \setminus K_{D,\boldsymbol{j}})^{\circ}$ due to the fact that $K_{D,\boldsymbol{j}}$ is closed. Hence we can find a ball $B_r(x) \subseteq \mathbb{R}^s$ centred at x with radius $r > 0$ which has empty intersection with $K_{D,\boldsymbol{j}}$. This implies $x \notin \tilde{K}_{n_k}$ for large $k \in \mathbb{N}$ which is a contradiction.

(ii) This follows directly from the Baire category theorem.

(iii) Since the attractors $K_{D,\boldsymbol{j}}$ are bounded, there exists $R \in \mathbb{R}$ such that $||x||_{\infty} < R$ for all $x \in K_{D,\boldsymbol{j}}$. Using that the integer shifts of $K_{D,\boldsymbol{j}}$ cover $\mathbb{R}^s$, it follows that $[-N+R, N-R]^s \subseteq K_{D,\boldsymbol{j}} + \{-N, \dots, N\}^s$ for all $N \in \mathbb{N}$, $N > R$. Therefore, $(2(N-R))^s \leq \lambda\left(K_{D,\boldsymbol{j}} + \{-N, \dots, N\}^s\right) \leq (2N)^s \cdot \lambda\left(K_{D,\boldsymbol{j}}\right)$ which implies $\frac{(N-R)^s}{(N+1)^s} \leq \lambda\left(K_{D,\boldsymbol{j}}\right)$ for all $N \in \mathbb{N}$, $N > R$. □

Example 2.1.17. We consider again the double dragon $K_{\mathbb{D}}$ from Example 2.1.10 *(ii)*, given by $M_{\mathbb{D}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $D_{\mathbb{D}} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Let $\tilde{K}_0 = [0,1]^2$. The first eight approximations $\tilde{K}_n$, $n \in \{0, \dots, 7\}$, of the attractor $K_{D_{\mathbb{D}}} = K_{\mathbb{D}}$, as defined in (2.1.7) with $Q = D_{\mathbb{D}}$, are plotted in Figure 2.3. △

Corollary 2.1.19 of Proposition 2.1.15 allows us to characterize attractors $K_{Q,\boldsymbol{j}}$ as fixed points of an iterated function system.

Definition 2.1.18. Let $\boldsymbol{j} \in \{1, \dots, J\}^{\mathbb{N}}$. For $r \in \mathbb{N}$ we define the *shifted sequence* $\boldsymbol{j}^{[r]}$ by

$$\boldsymbol{j}^{[r]} = (j_r, j_{r+1}, j_{r+2}, \cdots) = (j_{n+r-1})_{n\in\mathbb{N}}. \tag{2.1.9}$$

Corollary 2.1.19. *(i) If $\{M_j \in \mathbb{Z}^{s\times s} : j = 1, \dots, J\}$ is jointly expanding , $Q_j \subseteq \mathbb{Z}^s$, $j \in \{1, \dots, J\}$, finite and $\boldsymbol{j} \in \{1, \dots, J\}^{\mathbb{N}}$, then*

$$K_{Q,\boldsymbol{j}^{[r]}} = M_{j_r}^{-1}(Q_{j_r} + K_{Q,\boldsymbol{j}^{[r+1]}}), \quad r \in \mathbb{N}. \tag{2.1.10}$$

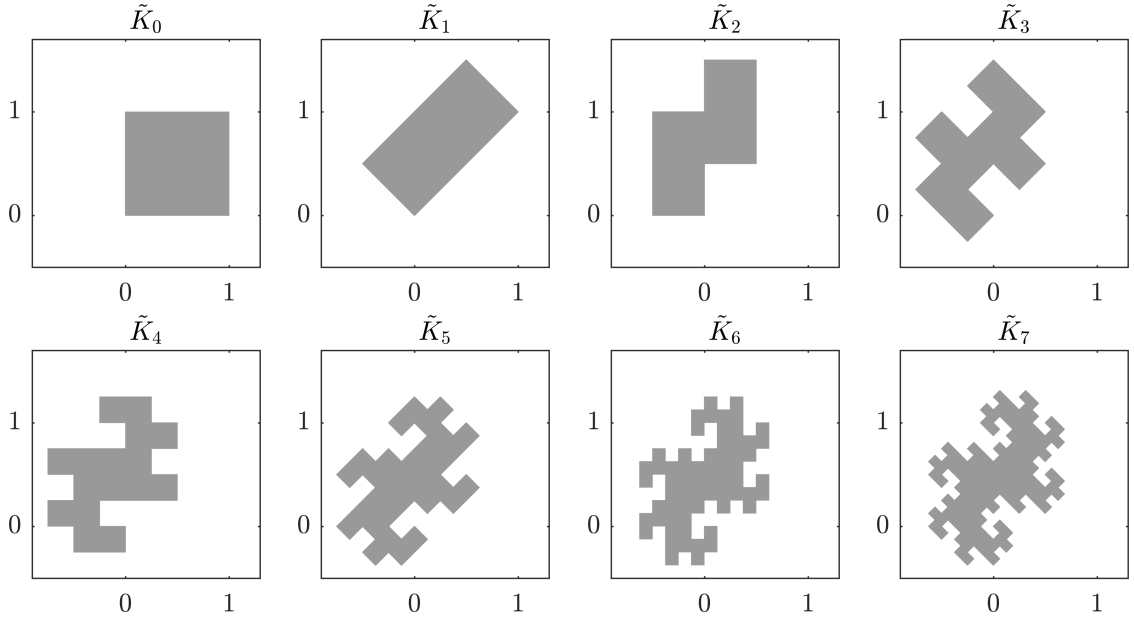


Figure 2.3: The first eight approximations of the double-dragon $K_{\mathbb{D}}$ from Example 2.1.17, starting with the unit cube

(*ii*) *Conversely, if* $(K_r)_{r\in\mathbb{N}}$, $K_r \subseteq \mathbb{R}^s$, *is a sequence of uniformly bounded, non-empty, compact subsets of* $\mathbb{R}^s$ *such that* $K_r = M_{j_r}^{-1}(Q_{j_r} + K_{r+1})$ *for all* $r \in \mathbb{N}$, *then*

$$K_r = K_{Q,\boldsymbol{j}^{[r]}}, \quad r \in \mathbb{N}.$$

Proof. (*i*) Since $\mathrm{cl}(A+B) = \mathrm{cl}(A) + \mathrm{cl}(B)$ for $A, B \subseteq \mathbb{R}^s$ compact, the claim follows by (2.1.3). Indeed,

$$\begin{aligned} K_{Q,\boldsymbol{j}^{[r]}} &= \mathrm{cl}(M_{j_r}^{-1}Q_r + M_{j_r}^{-1}M_{j_{r+1}}^{-1}Q_{j_{r+1}} + \ldots) \\ &= \mathrm{cl}(M_{j_r}^{-1}Q_r) + \mathrm{cl}(M_{j_r}^{-1}M_{j_{r+1}}^{-1}Q_{j_{r+1}} + \ldots) \\ &= M_{j_r}^{-1}Q_r + M_{j_r}^{-1}\,\mathrm{cl}(M_{j_{r+1}}^{-1}Q_{j_{r+1}} + \ldots) \\ &= M_{j_r}^{-1}Q_r + M_{j_r}^{-1}K_{Q,\boldsymbol{j}^{[r+1]}}. \end{aligned}$$

(*ii*) Applying $K_r = M_{j_r}^{-1}(Q_{j_r} + K_{r+1})$ iteratively we get

$$K_r = M_{j_r}^{-1}Q_{j_r} + \cdots + M_{j_r}^{-1}\cdots M_{j_n}^{-1}Q_{j_n} + M_{j_1}^{-1}\cdots M_{j_n}^{-1}K_n, \quad r \in \mathbb{N},\ n \in \mathbb{N}.$$

The compact sets K_r, $r \in \mathbb{N}$ are uniformly bounded and non-empty, and thus, by Proposition 2.1.15, $K_r = K_{Q,\boldsymbol{j}^{[r]}}$. □

Example 2.1.20. Let $s = 1$, $M_j \in \mathbb{N}$, $M_j \geq 2$, $D_j = \{0, \ldots, n-1\}$, for $j \in \{1, \ldots, J\}$. The attractor $K_{Q,\boldsymbol{j}^{[r]}}$ is the interval $[0,1]$ for any $\boldsymbol{j} \in \{1, \ldots, J\}^{\mathbb{N}}$, $r \in \mathbb{N}$. Indeed, the interval $[0,1]$ is a fixed point of the equations

$$K = M_j^{-1}(K + D_j), \quad j \in \{1, \ldots, J\},$$

and thus the claim follows by Corollary 2.1.19 (*ii*). △

The following corollary of Proposition 2.1.15 is useful for attractor calculus.

Corollary 2.1.21. *If* $\{M_j \in \mathbb{Z}^{s\times s} : j = 1,\dots,J\}$ *is jointly expanding,* $Q_j, Q'_j \subseteq \mathbb{Z}^s$, $j \in \{1,\dots,J\}$, *are finite and* $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$, *then*

$$K_{Q,\boldsymbol{j}} + K_{Q',\boldsymbol{j}} = K_{Q+Q',\boldsymbol{j}},$$

where $(Q+Q')_j = Q_j + Q'_j$.

Proof. For any $r \in \mathbb{N}$, by Corollary 2.1.19, we have

$$\begin{aligned} K_{Q,\boldsymbol{j}^{[r]}} + K_{Q',\boldsymbol{j}^{[r]}} &= M_{j_r}^{-1}(Q_{j_r} + K_{Q,\boldsymbol{j}^{[r+1]}}) + M_{j_r}^{-1}(Q'_{j_r} + K_{Q',\boldsymbol{j}^{[r+1]}}) \\ &= M_{j_r}^{-1}((Q_{j_r} + Q'_{j_r}) + (K_{Q,\boldsymbol{j}^{[r+1]}} + K_{Q',\boldsymbol{j}^{[r+1]}})). \end{aligned}$$

Therefore, the sets $K_{Q,\boldsymbol{j}^{[r]}} + K_{Q',\boldsymbol{j}^{[r]}}$, $r \in \mathbb{N}$, satisfy the assumptions of Corollary 2.1.19 (ii) and the claim follows. □

In particular, Corollary 2.1.21 shows that a shift of the sets Q_j only leads to a shift of the attractors $K_{Q,\boldsymbol{j}}$. Thus, we could settle for the standard choice that $0 \in Q_j$, although we never use this property.

Example 2.1.22. If $M{=}2$, $Q{=}\{0,1\}$, $Q'{=}\{1\}$, then $K_Q{=}[0,1]$, $K_{Q'}{=}\{1\}$, $K_{Q+Q'}{=}[1,2]$. △

2.2 Tiles

Most of the properties of stationary attractors transfer directly to the multiple case, but not all of them do. Even more, one desirable property does not even hold in general in the stationary case; namely that the integer shifts of attractors $K_{D,\boldsymbol{j}}$ corresponding to digits sets are essentially disjoint. A set with such a property is a *tile*, see Definition 2.2.1.

The current theory of the matrix approach that investigates properties of (stationary) subdivision schemes assumes that for the corresponding dilation matrix M, there exists a digit set D such that the attractor K_D is a tile. We show in Chapter 3 that this assumption is superfluous, and thus simplify the matrix approach for both the stationary and multiple case.

Definition 2.2.1. Let K be a measurable subset of $\mathbb{R}^s$. K is a *tile* if $K + \mathbb{Z}^s \simeq \mathbb{R}^s$ and $K \cap (K+\alpha) \simeq \emptyset$ for all $\alpha \in \mathbb{Z}^s \setminus \{0\}$. We say that K *tiles* $\mathbb{R}^s$.

Tiles have Lebesgue measure one. Given $K \subseteq \mathbb{R}^s$, measurable, such that $K + \mathbb{Z}^s \simeq \mathbb{R}^s$, then $K \cap (K+\alpha) \simeq \emptyset$ for all $\alpha \in \mathbb{Z}^s \setminus \{0\}$ if and only if $\lambda(K) = 1$. A proof for this standard argument can be found in [Gröchenig and Madych, 1992, Lemma 1].

Example 2.2.2. Tiles we have encountered so far are the interval $[0,1]$ from Example 2.1.20 and the double dragon $K_{\mathbb{DD}}$ from Example 2.1.10 (ii). △

The problem of the existence of pairs (M,D) of a dilation matrix M and corresponding digit set D, whose attractor K_D is a tile is partly solved in the stationary case. Lagarias and Wang [1995, 1996a, 1997] proved, if $s = 1,2,3$ or $|\det M| \geq s$, then there always exists a digit set D such that the attractor K_D is a tile. Potiopa [1997] showed that for the stationary case in dimension $s = 4$ for the dilation matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & -1 & 1 \end{bmatrix}$$

there exists no digit set, such that the corresponding attractor is a tile. For certain matrix families explicit digit sets are known such that the corresponding attractor is a tile [Gröchenig and Haas, 1994].

Furthermore, Gröchenig and Madych [1992] developed an algorithm which decides whether the attractor corresponding to a pair (M, D) is a tile or not.

For the multiple case, a natural problem arises: Given two jointly expanding dilation matrices $M_j \in \mathbb{Z}^{s\times s}$, $j \in \{1, 2\}$, and the corresponding digit sets D_j, whose stationary attractors K_{D_j} are tiles for $\mathbb{R}^s$. Does it follow that a multiple attractor $K_{D,\boldsymbol{j}}$ corresponding to $\{(M_1, D_1), (M_2, D_2)\}$ and arbitrary ordering $\boldsymbol{j} \in \{1, 2\}^{\mathbb{N}}$ is a tile? The answer is *no* as Example 2.2.3 shows. Moreover, two pairs (M_j, D_j), $j \in \{1, 2\}$, whose corresponding attractors are not tiles, can give rise to a multiple attractor which is a tile. This is shown in Example 2.2.4.

Example 2.2.3. Let $M_1 = \left[\begin{smallmatrix} 1 & 2 \\ -2 & -2 \end{smallmatrix}\right]$, $D_1 = \left\{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]\right\}$, $M_2 = \left[\begin{smallmatrix} -1 & 1 \\ -1 & -2 \end{smallmatrix}\right]$ and $D_2 = \left\{\left[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right\}$. The attractors K_{D_1} and K_{D_2} each tile $\mathbb{R}^2$, i.e. have area one. Nevertheless, the multiple attractor $K_{D_{12}}$ defined by alternating M_1 and M_2 is not a tile. This can be seen, by defining the stationary attractor corresponding to

$$M_{12} = M_2M_1 = \left[\begin{smallmatrix} -3 & -4 \\ 3 & 2 \end{smallmatrix}\right], \quad D_{12} = M_2D_1 + D_2 = \left\{\left[\begin{smallmatrix} -2 \\ -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -2 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right\},$$

and using the method described in [Gröchenig and Haas, 1994, (5), (6)], which we implemented in Matlab, or heuristically by looking at the plots of the attractor in Figure 2.4; the attractor $K_{D_{12}}$ obviously has area greater than one. △

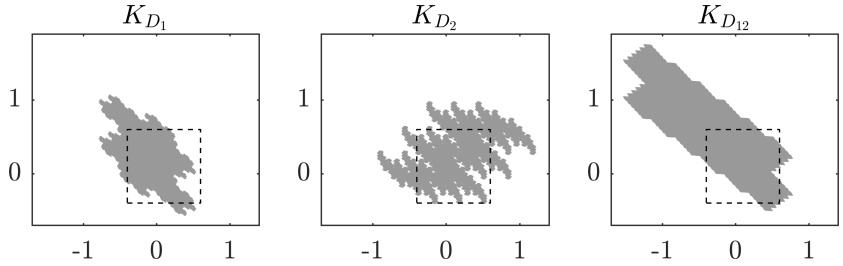


Figure 2.4: The attractors K_{D_1}, K_{D_2} and the multiple attractor $K_{D_{12}}$ as defined in Example 2.2.3. The first two attractors are tiles, whereas the third is not a tile. The black unit square is plotted to be able to compare the sizes of the attractors.

Similarly, there exists pairs of dilation matrices and digit sets whose attractors are not tiles, but a multiple scheme defined using these pairs corresponds to a tile.

Example 2.2.4. Let $M_3 = M_5 = 2$, $D_3 = \{0, 3\}$, $D_5 = \{0, 5\}$. Clearly $K_{D_3} = [0, 3]$ and $K_{D_5} = [0, 5]$. Quite surprisingly, the attractor $K_{D_{35}}$ arising from alternating M_3 and M_5, defined by $M_{35} = 4$, $D_{35} = \{0, 3, 10, 13\}$, is a tile. The attractor $K_{D_{35}}$ is plotted in Figure 2.5.

Again, we used our implementation of the algorithm in [Gröchenig and Haas, 1994] to check whether the attractors are tiles or not. △

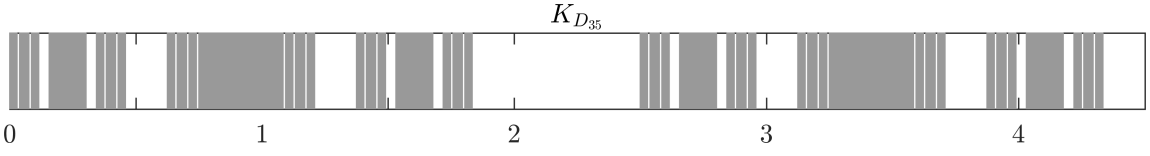


Figure 2.5: The attractor $K_{D_{35}}$ from Example 2.2.4, which is a tile.

Lagarias and Wang [1997, Theorem 1.1] proved that if $M \in \mathbb{Z}^{s\times s}$ is a dilation matrix and $D \subseteq \mathbb{Z}^s$ is a corresponding digit set, then

(*i*) $\lambda(K_D) \in \mathbb{N}$ and

(*ii*) there exists an invertible matrix $\Gamma \in \mathbb{Z}^{s\times s}$ such that $\Gamma\mathbb{Z}^s + K_D = \mathbb{R}^s$ and $(\gamma' + K_D) \cap (\gamma + K_D) \simeq \emptyset$ for all $\gamma, \gamma' \in \Gamma\mathbb{Z}^s$, $\gamma \neq \gamma'$.

Both statements are wrong in the multiple case in general, as Example 2.2.5 shows.

Example 2.2.5. For $M_1 = M_2 = 2$, $D_1 = \{0,1\}$, $D_2 = \{0,3\}$, the attractor $K_{D,\boldsymbol{j}}$ corresponding to the ordering $j_1 = j_2 = 1$, $j_n = 2$ for all $n \geq 3$, is the interval $[0, \frac{3}{2}]$, thus $\lambda(K_{D,\boldsymbol{j}}) \notin \mathbb{N}$. Clearly, there also does not exists a subset $X \subseteq \mathbb{Z}$ such that $X + K_{D,\boldsymbol{j}}$ covers $\mathbb{R}$ without essential overlap, thus (*ii*) does not hold either. To show that, we assume without loss of generality that $0 \in X$. To cover the point $\left\{\frac{7}{4}\right\}$ we need to have $1 \in X$, which on the other hand implies that $(0 + [0, \frac{3}{2}]) \cap (1 + [0, \frac{3}{2}]) \not\simeq \emptyset$. △

Example 2.2.5 shows that the Lebesgue measure of the attractors $K_{D,\boldsymbol{j}^{[r]}}$ may depend on r. Indeed, Lemma 2.2.6 shows that $\lambda\left(K_{Q,\boldsymbol{j}^{[r]}}\right)$, $r \in \mathbb{N}$, forms a bounded, monotone increasing sequence in r.

Proposition 2.2.6. *If $\{M_j \in \mathbb{Z}^{s\times s} : j = 1,\dots,J\}$ is jointly expanding with corresponding digits sets $D_j \simeq \mathbb{Z}^s/M_j\mathbb{Z}^s$ and $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$, then the Lebesgue measure of the attractors $K_{D,\boldsymbol{j}^{[r]}}$ is a bounded, monotone increasing sequence in $r \in \mathbb{N}$.*

Proof. Let $r \in \mathbb{N}$. Using that the dilation matrices M_j are invertible, $\#D_j = |\det M_j|$ for $j \in \{1,\dots,J\}$ and taking measure in Equation (2.1.10) we get

$$\begin{aligned} |\det M_{j_r}|\,\lambda\left(K_{D,\boldsymbol{j}^{[r]}}\right) &= \lambda\left(M_{j_r}K_{D,\boldsymbol{j}^{[r]}}\right) = \lambda\left(D_{j_r} + K_{D,\boldsymbol{j}^{[r+1]}}\right) \\ &\leq \#D_{j_r}\lambda\left(K_{D,\boldsymbol{j}^{[r+1]}}\right) = |\det M_{j_r}|\,\lambda\left(K_{D,\boldsymbol{j}^{[r+1]}}\right). \end{aligned} \tag{2.2.1}$$

Thus, the Lebesgue measures of the attractors are monotonically increasing with $r \in \mathbb{N}$. The sequence is bounded by Lemma 2.1.14. □

We failed to provide a concrete example of a multiple attractor whose area is strictly increasing in $r \in \mathbb{N}$.

Conjecture 2.2.7. *Let $M_1 = M_2 = 2$, $D_1 = \{0,1\}$, $D_2 = \{0,3\}$ and define $\boldsymbol{j} = (j_n)_n \in \{1,2\}^{\mathbb{N}}$ by*

$$j_n = \begin{cases} 1 & \text{if } n = m^2 \text{ for } m \in \mathbb{N} \text{ and} \\ 2 & \text{otherwise.} \end{cases}$$

We conjecture that $\lambda\left(K_{D,\boldsymbol{j}^{[r]}}\right)$ is strictly increasing in $r \in \mathbb{N}$.

In the stationary case the inequality in (2.2.1) becomes an equality. Using this, it is straightforward to show (see e.g. [Lagarias and Wang, 1996b, Theorem 1.1]) that the boundary of stationary attractors have Lebesgue measure zero. Whether the same holds for multiple attractors remains an open problem, worth to be investigated.

Remark 2.2.8. The function `checktile` [Mejstrik, 2018b] implements an algorithm by Gröchenig and Haas [1994], checking whether the attractor corresponding to a pair of a dilation matrix and corresponding digit set is a tile or not. A simple Matlab implementation of this function is listed in Section 2.3.

2.3 Appendix for Chapter 2

Algorithm 2.3.1. Implementation of the algorithm [Gröchenig and Haas, 1994] in Matlab. The algorithm checks whether the attractor corresponding to a pair of a dilation matrix and a corresponding digit set is a tile.

```
function [ flag, C, T ] = checktile( M, D )
% Computes if a pair dilation-matrix/digit-set generates a tile
% M: dilation matrix; D: digit set for M;
% flag: 1: tile, 0: no tile, NaN: Result needs to be checked by hand;
% C: Contact matrix; T: Contact vectors
% E.g.: M = [ 1 2; -1 1 ];  D = [ 0 0; 1 0; 2 0 ]'; [ flag, C, T ] = checktile( M, D );
    T = contactvectors( M, D );           %make contact vectors
    C = contactmatrix( M, D, T );         %make contact matrix
    rhoval = max(abs(eig(C))); detval = abs(det(M));
    if(rhoval<detval-.1); flag = 1; fprintf('(M,D) generates a tile.\n');
    elseif(rhoval>detval+.1); flag = 0; fprintf('(M,D) generates no tile.\n');
    else; flag = nan; fprintf('If rho(C)<abs(det(M)), then (M,D) generates a tile.\n');
    end;

function [ T ] = contactvectors( M, D )
% Computes the matrices \mathcal{T}_n and \mathcal{T} as described in
% Groechenig, Haas - Self-Similar Lattice Tilings, Thm 2.2 and Lem 4.5
    dim = size(M,1);
    T0 = [ eye(dim) -eye(dim) ]; %start-vectors for lattice \ZZ^2, T0=\mathcal{T}_0
    Tn{1} = T0;
    detM = abs(det(M));
    while(true)
        val = M\setplus(Tn{end},D,-D);
        idx = all(abs(val-round(val))<1/(2*detM),1);
        Tn{end+1} = unique([ round(val(:,idx)) Tn{end} ].','rows').';
                                  %The Tn are not necessarily contained in each other
        if(isequal(Tn{end},Tn{end-1}));
            T = unique([Tn{:}].','rows').';
            break;
        end;
    end
    idx = ismember(T.',zeros(1,dim),'rows');
    T(:,idx) =[]; %remove zero vector, see Def of \mathcal{T}_n^* before Lem 4.5

function [ C ] = contactmatrix( M, D, T ); %constructs contact matrix
    nT = size(T,2); C = zeros(nT,nT);
    for k = 1:nT; for l = 1:nT;
            val = intersect( setplus(M*T(:,k),D).', setplus(T(:,l),D).', 'rows').';
            C(k,l) = size(val,2);
    end; end;

function [ X ] = setplus( varargin )
% setplus(A,B) = { x=a+b : a in A, b in B}, Note: The function operates column wise
    sze = size(varargin,2);               %number of sets
    X = varargin{sze};                    %the output set
    for j = sze-1:-1:1                    %iterate through all sets
        A = varargin{j};                  %the set to be added
        X = repmat(A,1,size(X,2))+reshape(repmat(X,size(A,2),1),size(A,1),[]); %add set
        X = unique(X','rows')';           %remove duplicates
    end
```

3 Multiple subdivision

Four is infinity.

(Nira Dyn)

In this chapter we study the matrix approach in the setting of multiple subdivision. Multiple subdivision schemes, in contrast to stationary or non-stationary schemes, allow for level dependent subdivision weights and for level dependent dilation matrices. The latter property of multiple subdivision makes the standard definition of the transition matrices, crucial ingredient of the matrix approach in the stationary and non-stationary settings, inapplicable. We show how to avoid this obstacle and characterize the convergence of multiple subdivision schemes in terms of the joint spectral radius of certain square matrices derived from subdivision weights.

3.1 General definitions

The definition of subdivision operators associated to finite sets of finitely supported masks and jointly expanding dilation matrices is done analogously to the stationary or the non-stationary case.

Definition 3.1.1 ([Cavaretta, Dahmen and Micchelli, 1991, Equation (1.4)])**.** For a *mask* $a \in \ell_0(\mathbb{Z}^s)$ and a dilation matrix $M \in \mathbb{Z}^{s\times s}$ the *subdivision operator* $S : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ defined by the pair (a, M) is given by

$$Sc = \sum_{\beta\in\mathbb{Z}^s} a(\cdot - M\beta)c(\beta). \tag{3.1.1}$$

We identify the operator S defined by the pair (a, M) with the pair (a, M) itself.

The concept of multiple subdivision schemes was introduced by Sauer [2012].

Definition 3.1.2. Given masks $\{a_j \in \ell_0(\mathbb{Z}^s) : j = 1, \dots, J\}$ and jointly expanding matrices $\{M_j \in \mathbb{Z}^{s\times s} : j = 1, \dots, J\}$.

(*i*) We define the *finite set $\mathcal{S}$ of subdivision operators S_j* by

$$\mathcal{S} = \left\{S_j = (a_j, M_j) : a_j \in \ell_0(\mathbb{Z}^s),\ M_j \in \mathbb{Z}^{s\times s},\ j = 1, \dots, J\right\}.$$

(*ii*) A sequence $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$, $j_n \in \{1, \dots, J\}$, is called a *(multiple) subdivision scheme.*

Remark 3.1.3. The concept of multiple subdivision generalizes the stationary and non-stationary setting. Indeed, the set $\mathcal{S}^{\mathbb{N}}$ of all possible (multiple) subdivision schemes contains stationary subdivision schemes; the constant sequences $(S_j)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$, $n \in \mathbb{N}$, with $S_j \in \mathcal{S}$. The set $\mathcal{S}^{\mathbb{N}}$ also includes certain non-stationary subdivision schemes; the sequences $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ with the subdivision operators $S_{j_n} \in \mathcal{S}$ defined by the pairs (a_{j_n}, M).

By omitting the assumption that the matrices $\{M_j \in \mathbb{Z}^{s\times s} : j = 1, \dots, J\}$ are jointly expanding, the setting of multiple subdivision also includes Lane-Riesenfeld algorithms [Lane and Riesenfeld, 1980; Cashman, Hormann and Reif, 2013].

Given a multiple subdivision scheme $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ and starting data $c \in \ell(\mathbb{Z}^s)$, in the n^{th} iteration of the subdivision process we attach the values of the sequence $S_{j_n} \cdots S_{j_1} c$ to the grid $M_{j_1}^{-1} \cdots M_{j_n}^{-1} \mathbb{Z}^s$. Since the dilation matrices are jointly expanding, the grid $M_{j_1}^{-1} \cdots M_{j_n}^{-1} \mathbb{Z}^s$ gets finer and finer and eventually the sequence may be identified with a function defined on $\mathbb{R}^s$.

Definition 3.1.4. Let $\mathcal{S}$ be a finite set of subdivision operators and $\boldsymbol{j} = (j_n)_n \in \{1, \dots, J\}^{\mathbb{N}}$.

(*i*) We say a (multiple) subdivision scheme $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ is *convergent* if for every sequence $c \in \ell_\infty(\mathbb{Z}^s)$ there exists a function $g_{c,\boldsymbol{j}} \in C(\mathbb{R}^s)$ (which is non-zero for at least one sequence c) such that

$$\lim_{n\to\infty} \left|\left| g_{c,\boldsymbol{j}}(M_{j_1}^{-1} \cdots M_{j_n}^{-1} \cdot) - S_{j_n} \cdots S_{j_1} c(\cdot) \right|\right|_{\ell_\infty} = 0. \tag{3.1.2}$$

(*ii*) If the subdivision scheme $(S_{j_n})_{n\in\mathbb{N}}$ is convergent, we write

$$g_{c,\boldsymbol{j}} = \lim_{n\to\infty} S_{j_n} \cdots S_{j_1} c. \tag{3.1.3}$$

(*iii*) We say that $\mathcal{S}^{\mathbb{N}}$ is *convergent*, whenever every subdivision scheme in $\mathcal{S}^{\mathbb{N}}$ is convergent.

This definition coincides in the stationary and non-stationary cases with the definition of convergence of stationary subdivision schemes [Cavaretta, Dahmen and Micchelli, 1991, Definition 2.1] and non-stationary subdivision schemes [Dyn and Levin, 1995, Definition 1].

Example 3.1.5. Let $S_1 = (a_1, M_1)$, $S_2 = (a_2, M_2)$, with

$$a_1 = \frac{1}{4}\begin{bmatrix} \mathbf{-1} & -2 & 2 & 6 & 3 \end{bmatrix}^T, \quad M_1 = 2,$$
$$a_2 = \frac{1}{4}\begin{bmatrix} \mathbf{3} & 2 & 2 & 2 & -1 \end{bmatrix}^T, \quad M_2 = 2.$$

Using the theory of stationary subdivision [Daubechies and Lagarias, 1992b, Theorem 2.2] we can compute that the stationary subdivision schemes $(S_1)_{n\in\mathbb{N}}$ and $(S_2)_{n\in\mathbb{N}}$ both converge to functions with Hölder-regularity $\alpha_1 = \alpha_2 = \log_2(4/3) \simeq 0.4150$. Nevertheless, there exists a multiple scheme consisting of operators S_1 and S_2 which is not convergent; for example the periodic scheme $(S_2 S_1)_{n\in\mathbb{N}}$. The generated sequences of $(S_1)_{n\in\mathbb{N}}$, $(S_2)_{n\in\mathbb{N}}$ and $(S_1 S_2)_{n\in\mathbb{N}}$ for starting data $c = \delta$ after 10 iterations is plotted in Figure 3.1.

For we lack any theory yet, we proof the statements in this example in Section 3.6. △

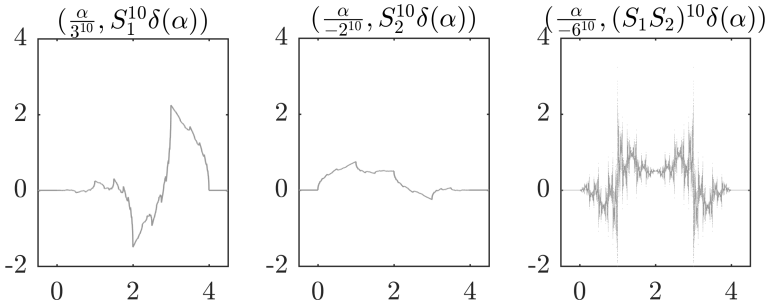


Figure 3.1: The subdivision schemes defined in Example 3.1.5 applied ten times to the δ-sequence. One can guess that the schemes $(S_1)_{n\in\mathbb{N}}$ and $(S_2)_{n\in\mathbb{N}}$ are convergent, whereas $(S_2 S_1)_{n\in\mathbb{N}}$ is not.

The necessary conditions, the sum rules of order one, for convergence of stationary subdivision schemes in $\mathcal{S}^{\mathbb{N}}$ are well known. If the stationary subdivision scheme corresponding to a subdivision operator $S = (a, M)$ is convergent, then $\sum_{\beta\in\mathbb{Z}^s} a(M\beta + \alpha) = 1$ for all $\alpha \in \mathbb{Z}^s$, see e.g. [Cavaretta, Dahmen and Micchelli, 1991; Daubechies and Lagarias, 1992a; Jetter and Plonka, 2001; Jia and Jiang, 2002]. Together with Definition 3.1.6 this immediately gives Lemma 3.1.7.

Definition 3.1.6 ([Strang and Fix, 1973, in statement of Theorem 1], [Jia, 1998, Equation (3.6)])**.** We say a subdivision operator $S = (a, M)$ satisfies

(*i*) *sum rules of order* 1 if for all $\alpha \in \mathbb{Z}^s$

$$\sum_{\beta\in\mathbb{Z}^s} a(M\beta + \alpha) = 1, \tag{3.1.4}$$

(*ii*) *sum rules of order* $k \in \mathbb{N}$, $k \geq 2$, if for all $\nu \in \mathbb{N}_0^s$, $0 \leq |\nu| \leq k-1$ and for all $\alpha, \alpha' \in \mathbb{Z}^s$

$$\sum_{\beta\in\mathbb{Z}^s} (M\beta + \alpha)^\nu a(M\beta + \alpha) = \sum_{\beta\in\mathbb{Z}^s} (M\beta + \alpha')^\nu a(M\beta + \alpha'). \tag{3.1.5}$$

Lemma 3.1.7. *If $\mathcal{S}^{\mathbb{N}}$ is convergent, then every stationary subdivision scheme defined by the pair (a_j, M_j), $j \in \{1, \ldots, J\}$ is convergent and its mask a_j satisfies the sum rules of order* 1*.*

The result of Lemma 3.1.7 gives rise to the following assumption.

Assumption 3.1.8 (S). *We assume that the masks a_j, $j \in \{1, \ldots, J\}$ satisfy sum rules of order 1.*

Using sum rules one can construct masks of subdivision schemes.

Example 3.1.9. We construct a univariate subdivision operator with dilation $M = 2$, whose mask a has support $\operatorname{supp} a = \{0, 1, 2, 3\}$ and satisfies sum rules of order 3. By (3.1.4) and (3.1.5) we get for $k = 1, 2, 3$, the following system of equations

$$\begin{aligned} a_1 + a_3 &= a_0 + a_2 = 1 \\ a_1 + 3a_3 &= 2a_2 \\ a_1 + 9a_3 &= 4a_2, \end{aligned}$$

which has the unique solution $a = \frac{1}{4}\begin{bmatrix}\mathbf{1} & 3 & 3 & 1\end{bmatrix}^T$. The basic limit function of the subdivision scheme corresponding to this subdivision operator is plotted in Figure 3.2. Note that sum rules alone are not sufficient for a subdivision scheme to be convergent, see Example 3.1.5. △

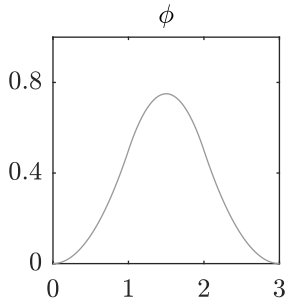


Figure 3.2: The basic limit function for the subdivision scheme from Example 3.1.9.

The limit function $g_{c,\boldsymbol{j}}$ in (3.1.3) with the starting sequence $c = \delta$ is called *basic limit function*. For multiple subdivision schemes, similarly to the non-stationary setting, every convergent scheme possesses a sequence of basic limit functions, which are mutually refinable and generated by certain (multiple) subdivision schemes $(S_{j_n})_{n\geq r} \in \mathcal{S}^{\mathbb{N}}$, $r \in \mathbb{N}$, related by the shift in the ordering of the corresponding subdivision operators.

Definition 3.1.10. For a (multiple) convergent subdivision scheme $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$, we define the sequence of *basic limit functions* by

$$\phi_{\boldsymbol{j}^{[r]}} = \lim_{n\to\infty} S_{j_{r+n}} \cdots S_{j_{r+1}} S_{j_r} \delta, \quad \delta(\alpha) = \begin{cases} 1, & \alpha = 0 \\ 0, & \text{otherwise} \end{cases}, \quad r \in \mathbb{N}. \tag{3.1.6}$$

If the scheme $(S_{j_n})_{n\in\mathbb{N}}$ is stationary, i.e. $S_{j_n} = S$ for all $n \in \mathbb{N}$, then $\phi_{\boldsymbol{j}^{[r]}} = \phi$ for all $r \in \mathbb{N}$.

Remark 3.1.11. Note that, since $\boldsymbol{j}^{[r]} \in \{1, \ldots, J\}^{\mathbb{N}}$, $r = 2, 3, \ldots$, the functions $\phi_{\boldsymbol{j}^{[r]}}$ by themselves are limits of subdivision schemes in $\mathcal{S}^{\mathbb{N}}$.

The proof of the mutual refinability of the basic limit functions $\phi_{\boldsymbol{j}^{[r]}}$, $r \in \mathbb{N}$, is analogous to the stationary or non-stationary setting. Lemma 3.1.12 and Proposition 3.1.13 are a direct generalization of [Cavaretta, Dahmen and Micchelli, 1991, Theorem 2.1]. Lemma 3.1.12 gives another constructive approximation of basic limit functions $\phi_{\boldsymbol{j}}$ than Definition 3.1.10, by refining functions instead of sequences.

Lemma 3.1.12. *Let $\psi_n \in C_0(\mathbb{R}^s)$, $n \in \mathbb{N}$, be a sequence of continuous, uniformly bounded functions, i.e. there exists $C_\psi > 0$, $K_\psi \subseteq \mathbb{R}^s$, compact, such that*

$$||\psi_n||_\infty \leq C_\psi < \infty, \quad n \in \mathbb{N},$$

with support

$$\operatorname{supp} \psi_n \subseteq K_\psi, \quad n \in \mathbb{N},$$

and such that

$$\sum_{\alpha\in\mathbb{Z}^s} \psi_n(x - \alpha) = 1, \quad x \in \mathbb{R}^s, \ n \in \mathbb{N}.$$

If $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ is a convergent subdivision scheme and $c \in \ell(\mathbb{Z}^s)$ then

$$g_{c,\boldsymbol{j}^{[r]}} = \lim_{n\to\infty} \sum_{\alpha\in\mathbb{Z}^s} c(\alpha) T_{j_r} \cdots T_{j_n} \psi_n(\cdot - \alpha), \quad r \in \mathbb{N}, \tag{3.1.7}$$

uniformly on compact subsets of $\mathbb{R}^s$, where

$$T_j \psi_n = \sum_{\alpha\in\mathbb{Z}^s} a_j(\alpha) \psi_n(M_j \cdot - \alpha), \quad j \in \{1, \ldots, J\}, \ n \in \mathbb{N}. \tag{3.1.8}$$

Proof. By Remark 3.1.11, we can assume without loss of generality that $r = 1$. By the definition of T_{j_n}, we get for $n \in \mathbb{N}$, $x \in \mathbb{R}^s$,

$$\begin{aligned} \sum_{\alpha\in\mathbb{Z}^s} c(\alpha) T_{j_n} \psi_n(x - \alpha) &= \sum_{\alpha\in\mathbb{Z}^s} c(\alpha) \sum_{\beta\in\mathbb{Z}^s} a_{j_n}(\beta) \psi_n(M_{j_n} x - M_{j_n}\alpha - \beta) \\ &= \sum_{\gamma\in\mathbb{Z}^s} \sum_{\alpha\in\mathbb{Z}^s} c(\alpha) a_{j_n}(\gamma - M_{j_n}\alpha) \psi_n(M_{j_n} x - \gamma) \\ &= \sum_{\gamma\in\mathbb{Z}^s} S_{j_n} c(\gamma) \psi_n(M_{j_n} x - \gamma). \end{aligned} \tag{3.1.9}$$

We can apply (3.1.9) iteratively to obtain for all $n \in \mathbb{N}$, $x \in \mathbb{R}^s$,

$$\sum_{\alpha \in \mathbb{Z}^s} c(\alpha) T_{j_1} \cdots T_{j_n} \psi_n(x - \alpha) = \sum_{\gamma \in \mathbb{Z}^s} S_{j_n} \cdots S_{j_1} c(\gamma) \psi_n(M_{j_n} \cdots M_{j_1} x - \gamma). \tag{3.1.10}$$

We show next that the right hand side of (3.1.10) converges uniformly on compact subsets to $g_{c,\boldsymbol{j}}$ in Definition 3.1.4. Let $\epsilon > 0$ and $K \subseteq \mathbb{R}^s$ compact. For $x \in K$, $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\Big| g_{c,\boldsymbol{j}}(x) - \sum_{\gamma \in \mathbb{Z}^s} S_{j_n} \cdots S_{j_1} c(\gamma) \psi_n(M_{j_n} \cdots M_{j_1} x - \gamma) \Big| = \\
&\quad = \sum_{\substack{\gamma \in \mathbb{Z}^s \\ M_{j_n} \cdots M_{j_1} x - \gamma \in K_\psi}} \Big| \big(g_{c,\boldsymbol{j}}(x) - S_{j_n} \cdots S_{j_1} c(\gamma) \big) \psi_n(M_{j_n} \cdots M_{j_1} x - \gamma) \Big| \\
&\quad \leq C_\psi \sum_{\substack{\gamma \in \mathbb{Z}^s \\ M_{j_n} \cdots M_{j_1} x - \gamma \in K_\psi}} \Big| g_{c,\boldsymbol{j}}(x) - S_{j_n} \cdots S_{j_1} c(\gamma) \Big| \\
&\quad \leq C_\psi \sum_{\substack{\gamma \in \mathbb{Z}^s \\ M_{j_n} \cdots M_{j_1} x - \gamma \in K_\psi}} \Big(\Big| g_{c,\boldsymbol{j}}(x) - g_{c,\boldsymbol{j}}(M_{j_1}^{-1} \cdots M_{j_n}^{-1} \gamma) \Big| + \Big| g_{c,\boldsymbol{j}}(M_{j_1}^{-1} \cdots M_{j_n}^{-1} \gamma) - S_{j_n} \cdots S_{j_1} c(\gamma) \Big| \Big).
\end{aligned} \tag{3.1.11}$$

Due to the uniform continuity of $g_{c,\boldsymbol{j}}$ on K, the fact that the dilation matrices are jointly expanding and that $\operatorname{supp} \psi_n \subseteq K_\psi$ is compact, we conclude that there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $M_{j_n} \cdots M_{j_1} x - \gamma \in K_\psi$,

$$\Big| g_{c,\boldsymbol{j}}(x) - g_{c,\boldsymbol{j}}(M_{j_1}^{-1} \cdots M_{j_n}^{-1} \gamma) \Big| < \epsilon.$$

Due to the convergence of the subdivision scheme, there exists $N_2 \in \mathbb{N}$ such that for $n > N_2$, $\gamma \in \mathbb{Z}^s$,

$$\Big| g_{c,\boldsymbol{j}}(M_{j_1}^{-1} \cdots M_{j_n}^{-1} \gamma) - S_{j_n} \cdots S_{j_1} c(\gamma) \Big| < \epsilon.$$

With $y = M_{j_n} \cdots M_{j_1} x \in \mathbb{R}^s$ being a shift for K_ψ, the number of terms in the sum

$$\sum_{\substack{\gamma \in \mathbb{Z}^s \\ M_{j_n} \cdots M_{j_1} x - \gamma \in K_\psi}}$$

in (3.1.11) can be bounded by $\# \{(y - K_\psi) \cap \mathbb{Z}^s\}$.

For $N \geq \max\{N_1, N_2\}$, using the triangle inequality and (3.1.10), we get that for $x \in K$, $n \in \mathbb{N}$,

$$\left| \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) T_{j_1} \cdots T_{j_n} \psi_n(x - \alpha) - g_{c,\boldsymbol{j}}(x) \right| < \max_{y \in \mathbb{R}^s} \# \{(y - K_\psi) \cap \mathbb{Z}^s\} \cdot C_\psi \cdot 2\epsilon.$$

Therefore,

$$\lim_{n \to \infty} \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) T_{j_1} \cdots T_{j_n} \psi_n(\cdot - \alpha) = g_{c,\boldsymbol{j}}. \qquad \square$$

Using Lemma 3.1.12 we prove the following properties of basic limit functions.

Proposition 3.1.13. *If $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ is a convergent subdivision scheme, then the following holds:*

(*i*) *The basic limit functions $\phi_{\boldsymbol{j}^{[r]}}$, $r \in \mathbb{N}$, are mutually refinable, i.e. they satisfy the system of refinement equations*

$$\phi_{\boldsymbol{j}^{[r]}}(x) = \sum_{\beta\in\mathbb{Z}^s} a_{j_r}(\beta)\phi_{\boldsymbol{j}^{[r+1]}}(M_{j_r}x - \beta), \quad x \in \mathbb{R}^s,\ r \in \mathbb{N}, \tag{3.1.12}$$

or, equivalently,

$$\phi_{\boldsymbol{j}^{[r]}} = T_{j_r}\phi_{\boldsymbol{j}^{[r+1]}}, \quad r \in \mathbb{N}, \tag{3.1.13}$$

where T_j is defined in (3.1.8).

(*ii*) *The basic limit function $\phi_{\boldsymbol{j}}$ is compactly supported, precisely,*

$$\operatorname{supp}\phi_{\boldsymbol{j}} \subseteq K_{\operatorname{supp} a,\boldsymbol{j}}, \quad \textit{with } (\operatorname{supp} a)_j = \operatorname{supp} a_j,\ j \in \{1,\dots,J\}.$$

(*iii*) *There exists $C_{\mathcal{S}} > 0$, depending only on $\mathcal{S}$, such that*

$$||\phi_{\boldsymbol{j}}||_\infty < C_{\mathcal{S}}.$$

(*iv*) *Any limit function of the multiple subdivision scheme corresponding to a starting sequence $c \in \ell(\mathbb{Z}^s)$ satisfies*

$$g_{c,\boldsymbol{j}} = \sum_{\alpha\in\mathbb{Z}^s} c(\alpha)\phi_{\boldsymbol{j}}(x-\alpha), \quad c \in \ell(\mathbb{Z}^s). \tag{3.1.14}$$

(*v*) *The basic limit functions $\phi_{\boldsymbol{j}}$ generates a partition of unity, i.e.*

$$\sum_{\alpha\in\mathbb{Z}^s} \phi_{\boldsymbol{j}}(x-\alpha) = 1, \quad x \in \mathbb{R}^s. \tag{3.1.15}$$

(*vi*) *The basic limit functions $\phi_{\boldsymbol{j}^{[r]}}$ are unique, i.e. they are the only continuous, uniformly bounded, compactly supported functions which satisfy* (3.1.12) *and* (3.1.15).

Proof. Let $\psi \in C_0(\mathbb{R}^s)$ be a continuous, compactly supported function, such that $\sum_{\alpha\in\mathbb{Z}^s}\psi(x-\alpha) = 1$ for all $x \in \mathbb{R}^s$.

(*i*) For $c = \delta$, by (3.1.6), we get $g_{c,\boldsymbol{j}^{[r]}} = \phi_{\boldsymbol{j}^{[r]}}$, $r \in \mathbb{N}$, and thus, by (3.1.7),

$$\lim_{n\to\infty} T_{j_{r+1}}\cdots T_{j_n}\psi = \phi_{\boldsymbol{j}^{[r+1]}}, \quad r \in \mathbb{N}.$$

Consequently, it follows that $\phi_{\boldsymbol{j}^{[r]}} = T_{j_r}\phi_{\boldsymbol{j}^{[r+1]}}$, $r \in \mathbb{N}$.

(*ii*) We show first by induction that

$$\operatorname{supp} S_{j_n}\cdots S_{j_1}\delta \subseteq M_{j_n}\cdots M_{j_2}\operatorname{supp} a_{j_1} + M_{j_n}\cdots M_{j_3}\operatorname{supp} a_{j_2} + \cdots + \operatorname{supp} a_{j_n}.$$

For $n = 1$ we have $S_{j_1}\delta = a_{j_1}$ which implies $\operatorname{supp} S_{j_1} = \operatorname{supp} a_{j_1}$. Now, assume that for $n \in \mathbb{N}$ we have shown that $\operatorname{supp} S_{j_n}\dots S_{j_1}\delta \subseteq M_{j_n}\cdots M_{j_2}\operatorname{supp} a_{j_1} + M_{j_n}\cdots M_{j_3}\operatorname{supp} a_{j_2} + \cdots + \operatorname{supp} a_{j_n}$. By the definition of the subdivision operator S_j, we get

$$S_{j_{n+1}}(S_{j_n}\cdots S_{j_1}\delta)(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{j_{n+1}}(\alpha - M_{j_{n+1}}\beta)S_{j_n}\dots S_{j_1}\delta(\beta),$$

which can only be non-zero if $\alpha - M_{j_{n+1}}\beta \in \operatorname{supp} a_{j_{n+1}}$ and $\beta \in \operatorname{supp} S_{j_n} \dots S_{j_1}\delta$. Therefore, $\alpha \in M_{j_{n+1}}\beta + \operatorname{supp} a_{j_{n+1}} \subseteq M_{j_{n+1}}M_{j_n}\cdots M_{j_2}\operatorname{supp} a_{j_1} + M_{j_{n+1}}M_{j_n}\cdots M_{j_3}\operatorname{supp} a_{j_2} + \cdots + M_{j_{n+1}}\operatorname{supp} a_{j_n} + \operatorname{supp} a_{j_{n+1}}$, which proves the claim.

Let $K_n = \{x \in \mathbb{R}^s : x = M_{j_1}^{-1}\cdots M_{j_n}^{-1}\alpha,\ \alpha \in \operatorname{supp} S_{j_n}\cdots S_{j_1}\delta\}$. Thus, $\lim_{n\to\infty} K_n = K_{\operatorname{supp} a, \boldsymbol{j}}$ in the Hausdorff metric. Therefore, using (3.1.2), we see that $\operatorname{supp}\phi_{\boldsymbol{j}} \subseteq K_{\operatorname{supp} a, \boldsymbol{j}}$.

(*iii*) Let $d : \{1,\dots,J\}^{\mathbb{N}}\times\{1,\dots,J\}^{\mathbb{N}} \to \mathbb{R}_+$ be metric define by $d(\boldsymbol{j}, \boldsymbol{j}') = \sum_{n=1}^{\infty} J^{-n}\,|j_n - j_n'|$. The space $\{1,\dots,J\}^{\mathbb{N}}$ with this metric is compact. By [Sauer, 2012, Corollary 1], the basic limit functions $\phi_{\boldsymbol{j}}$ depend continuously on $\boldsymbol{j}$ with respect to the metric d. Thus the supremum

$$C_{\mathcal{S}} = \sup_{\boldsymbol{j}\in\{1,\dots,J\}^{\mathbb{N}}} ||\phi_{\boldsymbol{j}}||_{\infty} < \infty$$

is attained and finite.

(*iv*) This follows directly from the linearity of the subdivision operators.

(*v*) For $c(\alpha) = 1$ for all $\alpha \in \mathbb{Z}^s$, it follows by Lemma 3.1.7 that $S_j c(\alpha) = 1$ for all $\alpha \in \mathbb{Z}^s$, $j \in \{1,\dots,J\}$. Thus, by (3.1.6), $g_{c,\boldsymbol{j}^{[r]}}(x) = 1$ for all $x \in \mathbb{R}^s$, $r \in \mathbb{N}$. Therefore, by (3.1.14),

$$1 = \sum_{\alpha\in\mathbb{Z}^s} \phi_{\boldsymbol{j}^{[r]}}(x-\alpha), \quad r \in \mathbb{N}.$$

(*vi*) Without loss of generality let $r = 1$. Assume there exists a sequence of uniformly bounded, continuous, compactly supported functions $\tilde{\phi}_n$, which satisfy (3.1.13) and (3.1.15). By (3.1.13) it follows that $\tilde{\phi}_1 = T_{j_1}\cdots T_{j_n}\tilde{\phi}_{n+1}$, $n \in \mathbb{N}$, and, by Lemma 3.1.12, it follows that $\lim T_{j_1}\cdots T_{j_n}\tilde{\phi}_{n+1} = \phi_{\boldsymbol{j}}$. Therefore, $\tilde{\phi}_1 = \phi_{\boldsymbol{j}^{[1]}}$. □

Example 3.1.14. Let $S_1 = (a_1, M_1)$ with $a_1 = \frac{1}{4}\begin{bmatrix}1 & 4 & 3\end{bmatrix}^T$, $M_1 = 2$, $S_2 = (a_2, M_2)$ with $a_2 = \frac{1}{4}\begin{bmatrix}1 & 3 & 3 & 1\end{bmatrix}^T$, $M_2 = 2$ and and $\boldsymbol{j} \in \{1,2\}^{\mathbb{N}}$ defined by $j_n = 1$ for n odd and $j_n = 2$ for n even. By Proposition 3.1.13 the basic limit functions $\phi_{\boldsymbol{j}^{[1]}}$ and $\phi_{\boldsymbol{j}^{[2]}}$ satisfy

$$\phi_{\boldsymbol{j}^{[1]}} = \frac{1}{4}\phi_{\boldsymbol{j}^{[2]}}(2\,\cdot) + \phi_{\boldsymbol{j}^{[2]}}(2\,\cdot - 1) + \frac{3}{4}\phi_{\boldsymbol{j}^{[2]}}(2\,\cdot - 2) \quad \text{and} \tag{3.1.16}$$

$$\phi_{\boldsymbol{j}^{[2]}} = \frac{1}{4}\phi_{\boldsymbol{j}^{[1]}}(2\,\cdot) + \frac{3}{4}\phi_{\boldsymbol{j}^{[1]}}(2\,\cdot - 1) + \frac{3}{4}\phi_{\boldsymbol{j}^{[1]}}(2\,\cdot - 2) + \frac{1}{4}\phi_{\boldsymbol{j}^{[1]}}(2\,\cdot - 3). \tag{3.1.17}$$

The two basic limit functions $\phi_{\boldsymbol{j}^{[1]}}$ and $\phi_{\boldsymbol{j}^{[2]}}$ can be seen in Figure 3.3 together with the dilated and scaled versions of $\phi_{\boldsymbol{j}^{[2]}}$ and $\phi_{\boldsymbol{j}^{[1]}}$ occurring in (3.1.16) and (3.1.17), respectively. △

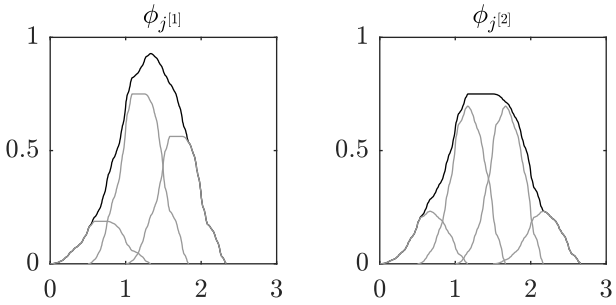


Figure 3.3: The basic limit functions from Example 3.1.14. The functions $\phi_{\boldsymbol{j}^{[1]}}$ and $\phi_{\boldsymbol{j}^{[2]}}$ are plotted in black. The scaled and dilated versions of $\phi_{\boldsymbol{j}^{[2]}}$ and $\phi_{\boldsymbol{j}^{[1]}}$ from (3.1.16) and (3.1.17) are plotted in grey.

Example 3.1.15 shows that for masks with negative entries $\operatorname{supp}\phi_{\boldsymbol{j}^{[r]}} \subsetneq K_{\operatorname{supp} a,\boldsymbol{j}^{[r]}}$ is indeed possible.

Example 3.1.15. Let $a = \frac{1}{2}\begin{bmatrix}\mathbf{1} & 2 & 1 & -1 & -2 & -1 & 1 & 2 & 1\end{bmatrix}^T$, $M = 2$. The corresponding basic limit function

$$\phi(x) = \begin{cases} x/3 & \text{if } x \in [0,1], \\ 2/3 - x/3 & \text{if } x \in [1,2], \\ -3/3 + x/3 & \text{if } x \in [3,4], \\ 5/3 - x/3 & \text{if } x \in [4,5], \\ -6/3 + x/3 & \text{if } x \in [6,7], \\ 8/3 - x/3 & \text{if } x \in [7,8] \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

has support $\operatorname{supp}\phi = [0,2] \cup [3,5] \cup [6,8]$ which is a strict subset of $K_{\operatorname{supp} a} = [0,8]$. The basic limit function is plotted in Figure 3.4. △

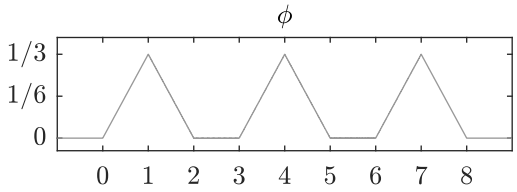


Figure 3.4: The basic limit function ϕ from Example 3.1.15.

Remark 3.1.16. The function `blf` [Mejstrik, 2018b] plots the basic limit function corresponding to multiple subdivision schemes.

3.2 Transition matrices and matrix refinement equations

By Proposition 3.1.13, for given $c \in \ell_\infty(\mathbb{Z}^s)$ the limit function $g_{c,\boldsymbol{j}}$ in (3.1.3) of the subdivision scheme $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ is a linear combination of the integer shifts of the corresponding basic limit function $\phi_{\boldsymbol{j}^{[1]}}$. Thus, the convergence analysis of $\mathcal{S}^{\mathbb{N}}$ is equivalent to the analysis of the corresponding basic limit functions. In Section 3.2, we show how to rewrite (3.1.12) to an equivalent vector-valued form, where the summation in (3.1.12) is replaced by a matrix vector multiplication. To do that we need to gain more insight into the structure of the supports of the basic limit functions.

Similarly to the stationary [Cavaretta, Dahmen and Micchelli, 1991] and non-stationary settings [Cohen and Dyn, 1996], there are two important ingredients of our construction: the transition operators and their common finite dimensional invariant subspaces. In Lemma 3.2.9 we provide an algorithm for the construction of a minimal, invariant subspace of the transition operators from Definition 3.2.1. The corresponding finite set Ω_C leads to transition matrices of minimal size, and thus is more suitable for numerical computations.

Definition 3.2.1. Let $j \in \{1,\dots,J\}$. For the subdivision mask $a_j \in \ell_0(\mathbb{Z}^s)$ and the dilation matrix $M_j \in \mathbb{Z}^{s\times s}$ with the digit set $D_j \simeq \mathbb{Z}^s/M_j\mathbb{Z}^s$, we define the *transition operator* $\mathcal{T}_{d,j} : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ by

$$\mathcal{T}_{d,j} v = \sum_{\beta\in\mathbb{Z}^s} v(\beta) a_j(M_j \cdot - \beta + d), \quad d \in D_j,\ v \in \ell(\mathbb{Z}^s). \tag{3.2.1}$$

The set of all transition operators is denoted by

$$\mathcal{T} = \{\mathcal{T}_{d,j} : d \in D_j,\ j = 1,\dots,J\}.$$

The result of Lemma 3.2.3 ensures the existence of a common finite dimensional invariant subspace of the transition operators in $\mathcal{T}$, which allows us to restrict the transition operators $\mathcal{T}_{d,j}$ to matrices.

Definition 3.2.2. Let $\Omega \subseteq \mathbb{Z}^s$. The set $\ell(\Omega) = \{v \in \ell(\mathbb{Z}^s) : \operatorname{supp} v \subseteq \Omega\}$ is the set of all sequences $v \in \ell(\mathbb{Z}^s)$ supported on Ω.

Lemma 3.2.3. *Let $\mathcal{S} = \{S_j : j = 1, \dots, J\}$ be a finite set of subdivision operators. There exists a finite set $\Omega_\mathbb{Z} \subseteq \mathbb{Z}^s$ such that $\ell(\Omega_\mathbb{Z})$ is invariant under all transition operators $\mathcal{T}_{d,j}$ in $\mathcal{T}$.*

Proof. We define the sets

$$\Omega_\mathbb{R} = \operatorname{cl} \bigcup_{\boldsymbol{j} \in \{1,\dots,J\}^\mathbb{N}} K_{\operatorname{supp} a - D, \boldsymbol{j}} \quad \text{and} \quad \Omega_\mathbb{Z} = \Omega_\mathbb{R} \cap \mathbb{Z}^s, \tag{3.2.2}$$

where $(\operatorname{supp} a - D)_j = \operatorname{supp} a_j - D_j$, $j \in \{1, \dots, J\}$. The set $\Omega_\mathbb{R}$ is bounded due to Lemma 2.1.14.

Let $d \in D_j$ for $j \in \{1, \dots, J\}$. We show that $\mathcal{T}_{d,j} : \ell(\Omega_\mathbb{Z}) \to \ell(\Omega_\mathbb{Z})$. Indeed, if $v \in \ell(\Omega_\mathbb{Z})$, then, by (3.2.1),

$$\begin{aligned}
\operatorname{supp}(\mathcal{T}_{d,j} v) &\subseteq M_j^{-1}(\operatorname{supp} a_j - d + \Omega_\mathbb{Z}) \cap \mathbb{Z}^s \subseteq \bigcup_{j \in \{1,\dots,J\}} M_j^{-1}(\operatorname{supp} a_j - D_j + \Omega_\mathbb{Z}) \cap \mathbb{Z}^s \\
&\subseteq \bigcup_{j \in \{1,\dots,J\}} M_j^{-1}\Big(\operatorname{supp} a_j - D_j + \operatorname{cl} \bigcup_{\boldsymbol{j} \in \{1,\dots,J\}^s} K_{\operatorname{supp} a - D, \boldsymbol{j}}\Big) \cap \mathbb{Z}^s \\
&\subseteq \operatorname{cl} \bigcup_{\boldsymbol{j} \in \{1,\dots,J\}^s} K_{\operatorname{supp} a - D, \boldsymbol{j}} \cap \mathbb{Z}^s = \Omega_\mathbb{Z},
\end{aligned}$$

where the last inclusion follows by $\operatorname{cl} A + \operatorname{cl} B \subseteq \operatorname{cl} A + B$ for $A, B \subseteq \mathbb{R}^s$. □

Example 3.2.4. We construct (numerically) the sets $\Omega_\mathbb{Z}$ and $\Omega_\mathbb{R}$ as defined in Lemma 3.2.3 for the multiple subdivision scheme introduced in [Cotronei, Ghisi, Rossini and Sauer, 2015] defined by $\mathcal{S} = \{(a_1, M_1), (a_2, M_2)\}$ with $a_1 = a_2 = \frac{1}{3}\begin{bmatrix}1 & 2 & \mathbf{3} & 2 & 1\end{bmatrix}$, $M_1 = \begin{bmatrix}1 & -1 \\ 1 & 2\end{bmatrix}$, $M_2 = \begin{bmatrix}2 & -1 \\ 1 & 2\end{bmatrix}$. The authors of [Cotronei, Ghisi, Rossini and Sauer, 2015] use this subdivision scheme to construct wavelets which can handle directional singularities.

The matrices M_1 and M_2 are jointly expanding, due to $\|M_{j_1}^{-1} M_{j_2}^{-1}\|_2 < 1$ for all $j_1, j_2 \in \{1, 2\}$. A numerical approximation of the set $\Omega_\mathbb{R}$ and $\Omega_\mathbb{Z}$ can be seen in Figure 3.5. The set $\Omega_\mathbb{R}$ is plotted in grey and the set $\Omega_\mathbb{Z}$ is marked with circles. Note that for points near the boundary (for example $\begin{bmatrix}-4 \\ -4\end{bmatrix}$) it is not clear whether they are in the set $\Omega_\mathbb{R}$ or not. Thus, for practical purposes, Lemma 3.2.3 cannot be used to determine an invariant set for the transition operators $\mathcal{T}_{d,j}$, due to the fractal nature of the attractors defined by (3.2.2). A more practical set introduced later is the set Ω_C constructed by Algorithm 3.2.9; its elements are marked with $\times$. △

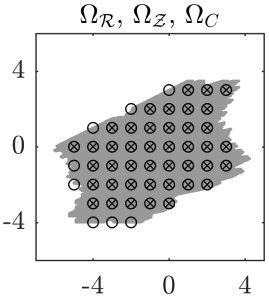


Figure 3.5: The sets $\Omega_\mathbb{R}$ (in grey), $\Omega_\mathbb{Z}$ (marked with $\circ$) and Ω_C (marked with $\times$) from Example 3.2.4.

Remark 3.2.5. The image on page 6 (as well as the image in Figure 3.5) are attractors defined by (3.2.2). These sets can be plotted using the function `tile` with the option `'supertile'`.

The result of Lemma 3.2.3 allows us to associate each transition operator $\mathcal{T}_{d,j}$ with a certain square matrix.

Definition 3.2.6. Let $\Omega \subseteq \mathbb{Z}^s$ be finite and such that $\ell(\Omega)$ is $\mathcal{T}$ invariant. For the operators in $\mathcal{T}$ we define the *transition matrices* by

$$T_{d,j,\Omega} = \Big[a_j(M_j\alpha - \beta + d)\Big]_{\alpha,\beta\in\Omega}, \quad d \in D_j,\ j \in \{1,\dots,J\}\,. \tag{3.2.3}$$

Remark 3.2.7. In the rest of the paper we use two other sets $\Omega \subseteq \mathbb{Z}^s$ such that $\ell(\Omega)$ is invariant under all operators in $\mathcal{T}$: the set Ω_C from Algorithm 3.2.9 for numerical computations and the larger set Ω_V from Proposition 3.4.21 for the theoretical analysis in Section 3.4.

The sum rules of order one for the masks a_j, Assumption **S** 3.1.8, become conditions on the spectral properties of the transition matrices.

Lemma 3.2.8. *Let $\mathcal{S}$ be a finite set of subdivision operators whose masks satisfy sum rules of order 1, $j \in \{1,\dots,J\}$, $d \in D$ and define $\mathbf{1} = \begin{bmatrix}1 & \dots & 1\end{bmatrix} \in \mathbb{R}^{1\times s}$ to be the row-vector of all ones of length s.*

(i) If $\mathcal{T}_{d,j} : \ell(\Omega) \to \ell(\Omega)$, $\Omega \subseteq \mathbb{Z}^s$, then the transition matrix $T_{d,j,\Omega}$ satisfies $\mathbf{1}\,T_{d,j,\Omega} = \mathbf{1}$.

(ii) If all entries of $T_{d,j,\Omega}$ are non-negative, then $\mathbf{1}\,T_{d,j,\Omega} = \mathbf{1}$ implies $\mathcal{T}_{d,j} : \ell(\Omega) \to \ell(\Omega)$.

Proof. *(i)* Invariance of $\ell(\Omega)$ under $\mathcal{T}_{d,j}$, $d \in D_j$, $j \in \{1,\dots,J\}$, implies, by Definition 3.2.1, that $a_j(M_j\alpha - \beta + d) = 0$, whenever $\alpha \notin \Omega$ and $\beta \in \Omega$. Since the masks a_j satisfy sum rules of order 1, we conclude that the entries in each column of the corresponding transition matrix $T_{d,j,\Omega}$ sum up to one, since

$$1 = \sum_{\alpha\in\mathbb{Z}^s} a_j(M_j\alpha - \beta + d) = \sum_{\alpha\in\Omega} a_j(M_j\alpha - \beta + d), \quad \beta \in \Omega.$$

(ii) Assume that $a_j(\alpha) \geq 0$ for all $\alpha \in \mathbb{Z}^s$, $j \in \{1,\dots,J\}$. Due to $\mathbf{1}\,T_{d,j,\Omega} = \mathbf{1}$, $d \in D_j$, we get

$$1 = \sum_{\alpha\in\Omega} a_j(M_j\alpha - \beta + d), \quad \beta \in \Omega.$$

The masks a_j satisfy sum rules of order one, which implies that $a_j(M_j\alpha - \beta + d) = 0$ for all $\alpha \notin \Omega$, $\beta \in \Omega$. Thus, $\mathcal{T}_{d,j} : \ell(\Omega) \to \ell(\Omega)$. □

For numerical computations of the joint spectral radius, the approximations of $\Omega_{\mathbb{Z}}$ defined in Lemma 3.2.3 are of no practical use. The straightforward observation that, without loss of generality, $0 \in \Omega_{\mathbb{R}}$ leads to Algorithm 3.2.9 for explicit computation of $\Omega_C \subseteq \Omega_{\mathbb{Z}}$ with desired invariance properties as in Lemma 3.2.8.

Algorithm 3.2.9 (Invariant Omega algorithm)**.**

$\Omega_0 \coloneqq \{0\}\,,\ n \coloneqq 0$
repeat
  $n \coloneqq n+1$
  $\Omega_n \coloneqq \Omega_{n-1}$
  for $j \in \{1,\dots,J\}$ **do**
    $\Omega_{n,j} \coloneqq \left(M_j^{-1}(\operatorname{supp} a_j + \Omega_n - D_j)\right) \cap \mathbb{Z}^s$
    $\Omega_n \coloneqq \Omega_n \cup \Omega_{n,j}$
  end
until $\Omega_n = \Omega_{n-1}$
$\Omega_C := \Omega_i$

Lemma 3.2.10. *Given a finite set of subdivision operators $\mathcal{S}$.*

(*i*) *Algorithm 3.2.9 constructs a finite set $\Omega_C \subseteq \mathbb{Z}^s$ such that $\ell(\Omega_C)$ is $\mathcal{T}$ invariant.*

(*ii*) *Ω_C is the smallest set $\Omega \subseteq \mathbb{Z}^s$ by inclusion with $0 \in \Omega$, such that $\ell(\Omega)$ is $\mathcal{T}$ invariant.*

Proof. (*i*) We show that the algorithm terminates after finitely many steps. More precisely, we show, by induction on $n \in \mathbb{N}$, that the sets $(\Omega_n)_{n=0}^N$, $N \in \mathbb{N}$, are increasing, nested subsets of a modified version of the finite set $\Omega_{\mathbb{Z}}$ determined in Lemma 3.2.3. Therefore, N is finite.

By assuming that $0 \in D_j$, $0 \in \operatorname{supp} a_j$, $j \in \{1,\dots,J\}$, by the uniqueness of attractors 2.1.19 (*ii*) and the fact that $\{0\} \subseteq M_j^{-1}(\operatorname{supp} a_j + \{0\} - D_j)$, $j \in \{1,\dots,J\}$, it follows that $\Omega_0 = \{0\} \subseteq \Omega_{\mathbb{Z}} \subseteq \Omega_{\mathbb{R}}$, where $\Omega_{\mathbb{R}}$ is defined by (3.2.2). This implies, there exists a set $\Omega_{\mathbb{Z}} \subseteq \mathbb{Z}^s$ such that $0 \in \Omega_{\mathbb{Z}}$ and $\ell(\Omega_{\mathbb{Z}})$ is $\mathcal{T}$ invariant.

Assume that $\Omega_n \subseteq \Omega_{\mathbb{Z}}$, $n \le N$. By Lemma 3.2.3, $\Omega_{\mathbb{Z}}$ is invariant under all operators $M_j^{-1}(\operatorname{supp} a_j + \,\cdot\, - D)$, and thus $\Omega_{n,j} \subseteq \Omega_{\mathbb{Z}}$ for all $j \in \{1,\dots,J\}$. Therefore, $\Omega_n = \Omega_{n-1} \cup \left(\bigcup_{j\in\{1,\dots,J\}} \Omega_{n,j}\right) \subseteq \Omega_{\mathbb{Z}}$. Due to the stopping criterion, we get increasing, nested sets $\Omega_0 \subsetneqq \Omega_1 \subsetneqq \Omega_2 \subsetneqq \cdots \subsetneqq \Omega_N \subseteq \Omega_{\mathbb{Z}}$. Moreover, for $\Omega_C = \Omega_N = \Omega_{N-1}$, due to (3.2.1), we get $\mathcal{T}_d : \ell(\Omega_{N-1}) \to \ell(\Omega_N)$, $d \in D_j$, $j \in \{1,\dots,J\}$. Therefore, the claim follows.

(*ii*) Any set $\Omega \subseteq \mathbb{Z}^s$, with $0 \in \Omega$ and which is $\mathcal{T}$ invariant is a superset of Ω_C by construction of Ω_C. □

Remark 3.2.11. The choice of $\Omega_0 = \{0\}$ in Algorithm 3.2.9 is not crucial. Algorithm 3.2.9 constructs a finite set $\Omega \subseteq \mathbb{Z}^s$, such that $\ell(\Omega)$ is $\mathcal{T}$ invariant, with any finite, non-empty, starting set $\Omega_0 \subseteq \mathbb{Z}^s$.

Remark 3.2.12. The function `constructOmega` [Mejstrik, 2018b] is an implementation of Algorithm 3.2.9. A simple implementation of Algorithm 3.2.9 for Matlab is given in Section 3.6.

In some cases, the sets $\Omega_{\mathbb{Z}}$ defined in Lemma 3.2.3 and Ω_C constructed by Algorithm 3.2.9 coincide.

Example 3.2.13. (*i*) For $M = 2$, $D = \{0,1\}$ and $\operatorname{supp} a = \{0,1,2,3\}$, Algorithm 3.2.9, generates $\Omega_C = \{0,1,2\}$ which is a strict subset of $\Omega_{\mathbb{Z}}$. Indeed, $\Omega_{\mathbb{R}} = M^{-1}(\operatorname{supp} a - D) + M^{-2}(\operatorname{supp} a - D) + \cdots = [-1,3]$, and thus $\Omega_{\mathbb{Z}} = \{-1,0,1,3\}$.

(*ii*) Given $M = -2$, $D = \{-1,0\}$ and mask a with support $\operatorname{supp} a = \{0,1,2,3\}$. By (3.2.2) $\Omega_{\mathbb{R}} = K_{\operatorname{supp} a - D, \boldsymbol{j}} = [-\frac{7}{3}, \frac{2}{3}]$, and thus $\Omega_{\mathbb{Z}} = \{-2,-1,0\}$. Algorithm 3.2.9 produces the same set. △

3.3 Support of basic limit functions

Similarly to the stationary and non-stationary settings, the supports of the basic limit functions $(\phi_{\boldsymbol{j}^{[r]}})_{r\in\mathbb{N}}$ can be covered by the integer shifts of the corresponding attractors $K_{D,\boldsymbol{j}^{[r]}}$ in Definition 2.1.5, i.e.

$$\operatorname{supp}\phi_{\boldsymbol{j}^{[r]}} \subseteq \Omega_{\mathbb{Z}} + K_{D,\boldsymbol{j}^{[r]}}, \tag{3.3.1}$$

see Lemma 3.3.1. This leads to a standard matrix form of the refinement equations used for analysing the existence and regularity of refinable functions in the stationary and multiple settings. Nevertheless, as already mentioned, even in the stationary setting the set $\Omega_{\mathbb{Z}}$ cannot be computed in general. Theorem 3.3.2 improves the result of Lemma 3.3.1, showing that any finite set $\Omega \subseteq \mathbb{Z}^s$, such that $\ell(\Omega)$ is $\mathcal{T}$ invariant, can replace $\Omega_{\mathbb{Z}}$ in (3.3.1).

Lemma 3.3.1 is a direct generalization of [Cabrelli, Heil and Molter, 2004, Proposition 2.7].

Lemma 3.3.1. *Let $\mathcal{S} = \{S_j : j = 1,\dots,J\}$ be a finite set of subdivision operators and $\Omega_{\mathbb{Z}} \subseteq \mathbb{Z}^s$ defined by* (3.2.2)*. If the subdivision scheme $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ is convergent, then $\operatorname{supp}\phi_{\boldsymbol{j}^{[r]}} \subseteq \Omega_{\mathbb{Z}} + K_{D,\boldsymbol{j}^{[r]}}$ for all $r \in \mathbb{N}$.*

Proof. Without loss of generality we assume $r = 1$, i.e. $\boldsymbol{j} = \boldsymbol{j}^{[1]}$. Let $x \in \operatorname{supp}\phi_{\boldsymbol{j}}$. By Lemma 3.1.13 (ii), $x \in K_{\operatorname{supp} a,\boldsymbol{j}}$, and thus there exist $\alpha_n \in \operatorname{supp} a_{j_n}$, $n \in \mathbb{N}$ such that $x = .\alpha_1\alpha_2\dots$. Furthermore, since integer shifts of $K_{D,\boldsymbol{j}}$ cover $\mathbb{R}^s$, there exists $\omega \in \mathbb{Z}^s$ and $d_n \in D_{j_n}$, $n \in \mathbb{N}$ such that $x = \omega + .d_1d_2\dots$, or equivalently, $\omega = .\alpha_1\alpha_2\dots - d_1d_2\dots$.

Therefore, by Corollary 2.1.21 and the definition of $\Omega_{\mathbb{R}}$ in (3.2.2), $\omega \in \Omega_{\mathbb{R}} \cap \mathbb{Z}^s = \Omega_{\mathbb{Z}}$ and $x \in \Omega_{\mathbb{Z}} + K_{D,\boldsymbol{j}}$. □

For practical purposes it is useful to replace $\Omega_{\mathbb{Z}}$ in Lemma 3.3.1 by an arbitrary set $\Omega \subseteq \mathbb{Z}^s$ such that $\ell(\Omega)$ is $\mathcal{T}$ invariant.

Theorem 3.3.2. *If $\mathcal{S}^{\mathbb{N}}$ is convergent and $\Omega \subseteq \mathbb{Z}^s$ finite such that $\ell(\Omega)$ is $\mathcal{T}$ invariant and $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$, then $\operatorname{supp}\phi_{\boldsymbol{j}^{[r]}} \subseteq \Omega + K_{D,\boldsymbol{j}^{[r]}}$ for all $r \in \mathbb{N}$.*

Proof. We prove the claim for the stationary case first. Let $\psi \in C_0(\mathbb{R}^s)$, $\operatorname{supp}\psi \subseteq \Omega + K_D$. Define, as in Proposition 3.1.13,

$$T\psi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\psi(Mx-\alpha), \quad x \in \mathbb{R}^s.$$

Since $\ell(\Omega)$ is $\mathcal{T}_d$ invariant for all $\mathcal{T}_d \in \mathcal{T}$, $d \in D$, it follows that $\operatorname{supp} T\psi \subseteq \Omega + K_D$. Indeed, if $\omega \in \mathbb{Z}^s$, $x = .d_1d_2\dots \in K_D$, $d_n \in D$, then

$$\begin{aligned} T\psi(\omega+x) &= \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\psi(M(\omega+x)-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} a(M\omega-\alpha+d_1)\psi(Mx-d_1+\alpha) \\ &= \sum_{\alpha\in\Omega} a(M\omega-\alpha+d_1)\psi(Mx-d_1+\alpha), \end{aligned} \tag{3.3.2}$$

due to $\operatorname{supp}\psi \subseteq \Omega + K_D$ and $Mx - d_1 \in K_D$. By (3.1.7), $T^n\psi \to \phi$ as $n \to \infty$, where ϕ is the basic limit function. We conclude that $\operatorname{supp}\phi \subseteq \Omega + K_D$.

We now show that $\phi_{\boldsymbol{j}} \subseteq K_{D,\boldsymbol{j}} + \Omega$ in the multiple case. For $N \in \mathbb{N}$ define the periodic sequence $\boldsymbol{n}(N) \in \{1,\dots,J\}^{\mathbb{N}}$ by

$$\boldsymbol{n}(N) = (j_1,\dots,j_N,j_1,\dots,j_N,j_1,\dots).$$

Note that $(S_{n_m})_{m\in\mathbb{N}} = (S_{j_n}\cdots S_{j_1})_{m\in\mathbb{N}}$ is a stationary subdivision scheme. Thus, by the first part of the proof $\operatorname{supp}\phi_{\boldsymbol{n}(N)} \subseteq \Omega + K_{D,\boldsymbol{n}(N)}$ for all $N \in \mathbb{N}$. By Proposition 2.1.15, since the sets

$K_{\operatorname{supp} a,\boldsymbol{n}(N)}$ and $K_{D,\boldsymbol{n}(N)}$ are uniformly bounded, $K_{\operatorname{supp} a,\boldsymbol{n}(N)} \to K_{\operatorname{supp} a,\boldsymbol{j}}$ and $\Omega + K_{D,\boldsymbol{n}(N)} \to \Omega + K_{D,\boldsymbol{j}}$ in the Hausdorff metric, as $N \to \infty$, since Ω does not depend on N. Therefore, $\operatorname{supp} \phi_{\boldsymbol{j}} \subseteq \Omega + K_{D,\boldsymbol{j}}$. □

Example 3.3.3. If $a = \begin{bmatrix}\mathbf{1} & 0 & 0 & 1\end{bmatrix}^T$, $M = 2$, then both $\Omega_1 = \{0\}$ and $\Omega_2 = \{-2,-1,1,2\}$ are minimal sets in the sense that there exists no set $\tilde{\Omega} \subsetneqq \Omega_i$, $i \in \{1,2\}$, such that $\ell(\tilde{\Omega})$ is invariant under all transition matrices. Note that the scheme (a, M) is not convergent which is easily seen by the fact that $S^n\delta(\alpha) = 0$ for all $\alpha \notin 3\mathbb{Z}$, Indeed, from this it would follow that the basic limit function is the zero function, which contradicts the assumption that the subdivision scheme is convergent. △

Under the additional assumption of convergence, all our numerical examples lead us to Conjecture 3.3.4.

Conjecture 3.3.4. *If $\mathcal{S}^{\mathbb{N}}$ is convergent, then there exists a unique, with respect to inclusion, minimal set $\Omega \subseteq \mathbb{Z}^s$ such that $\ell(\Omega)$ is $\mathcal{T}$ invariant.*

An immediate consequence of Theorem 3.3.2 and the mutual refinability of the basic limit functions is Corollary 3.3.5, which allows to compute function values of basic limit functions for certain subdivision schemes exactly, see e.g. [Daubechies and Lagarias, 1992b, Equation (2.6)] for the stationary case.

Corollary 3.3.5. *If $\mathcal{S}^{\mathbb{N}}$ is convergent and $\Omega \subseteq \mathbb{Z}^s$ is finite such that $\ell(\Omega)$ is $\mathcal{T}$ invariant and $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$, then for any $x = .d_r d_{r+1} \ldots \in K_{D,\boldsymbol{j}^{[r]}}$, $d_n \in D_{j_n}$, $r \in \mathbb{N}$,*

$$\left[\phi_{\boldsymbol{j}^{[r]}}(.d_r d_{r+1} \ldots + \omega)\right]_{\omega\in\Omega} = T_{d_r,j_r,\Omega}\left[\phi_{\boldsymbol{j}[r+1]}(.d_{r+1}d_{r+2}\ldots + \omega)\right]_{\omega\in\Omega}, \tag{3.3.3}$$

where $T_{d_r,j_r,\Omega}$ is defined in (3.2.3).

The following example shows how to compute the exact function values of the basic limit functions at rational points in $\mathbb{R}^s$ of all multiple subdivision schemes whose corresponding sequence $\boldsymbol{j}$ is eventually periodic.

Example 3.3.6. Let $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ be the multiple subdivision scheme with masks a_j, jointly expanding dilation matrices M_j, $j \in \{1,\dots,J\}$ with corresponding digit sets $D_j \simeq \mathbb{Z}^s/M\mathbb{Z}^s$, $j \in \{1,\dots,J\}$ and $\boldsymbol{j} \in \{1,\dots,J\}^{\mathbb{N}}$ eventually periodic, i.e. $\boldsymbol{j} = (j_1,\dots,j_k,\overline{j_{k+1},\dots,j_l})$, $l,k \in \mathbb{N}$, where $\overline{j_{k+1}\dots j_l}$ denotes the periodic part of $\boldsymbol{j}$. Furthermore, let $z \in \operatorname{supp}\phi_{\boldsymbol{j}} \cap \mathbb{Q}^s$.

Let Ω_C be the set constructed with Algorithm 3.2.9. Since $\operatorname{supp}\phi_{\boldsymbol{j}} \subseteq K_{D,\boldsymbol{j}} + \Omega_C$, there exists $x \in K_{D,\boldsymbol{j}}$ and $\tilde{\omega} \in \Omega_C$, such that $z = x + \tilde{\omega}$. Note that, since $x \in \mathbb{Q}^s$ and $\boldsymbol{j}$ is eventually periodic, the expansion of x is eventually periodic, and thus $x = .d_1\dots d_m\overline{d_{m+1}\dots d_n}$, $d_i \in D_{j_i}$. By (3.3.3) and using the periodicity of the digits d_i of x, we obtain

$$\left[\phi(.d_{m+1}d_{m+2}\ldots+\omega)\right]_{\omega\in\Omega_C} = T_{d_{m+1},j_{m+1}}\cdots T_{d_n,j_n}\left[\phi(.d_{m+1}d_{m+2}\ldots+\omega)\right]_{\omega\in\Omega_C}.$$

To obtain the function values $\left[\phi(.d_{m+1}d_{m+2}\ldots+\omega)\right]_{\omega\in\Omega_C}$, we have to compute an eigenvector $v \in \mathbb{R}^{\Omega_C}$ of $T_{d_{m+1},j_{m+1}}\cdots T_{d_n,j_n}$ to the eigenvalue 1 which exists by Lemma 3.2.8. By the uniqueness of the basic limit functions, Proposition 3.1.13 (vi), this eigenvalue is simple. By (3.1.15), we have to normalize the eigenvector v such that $\sum_{\omega\in\Omega_C} v_\omega = 1$. Finally, the value $\phi(z)$ is given by $T_{d_1,j_1}\cdots T_{d_m,j_m}v|_{\tilde{\omega}}$.

We present an example for Matlab for computing function values of basic limit functions. For the sake of simplicity and clarity we use a stationary subdivision scheme. Let `a=[.5 1 .5]`, `M=[1 1;-1 1]` and define the subdivision operator `S=getS({a,M})`. The function `getS` computes

the standard digit set given by (2.1.5) which we can view with `D=S{3}` and which is given by `d1=[0;0]` and `d2=[1;0]`. We next construct the set `OmC=constructOmega(S)`, given by

$$\texttt{OmC} = \left\{ \begin{bmatrix}-3\\-1\end{bmatrix}, \begin{bmatrix}-3\\0\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}, \begin{bmatrix}-2\\-1\end{bmatrix}, \begin{bmatrix}-2\\0\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}-1\\-2\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\-2\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\},$$

and plot the set `OmRR=tile(getS(S,'OmegaRR'))` from (3.2.2), the attractor `KD=tile(getS(S))` and the support of the basic limit function `KA=tile(getS(S,'supp'))`, for recreational purposes. We choose the point `z=[-.2;.4]` and compute the values of `wt`, `x` and its digit expansion `oo` by `[oo,~,x]=num2ordering(S,z,'check')`. Using the notation from above, the first cell array in `oo` is the part $.d_1 \dots d_m$ and the second cell array in `oo` is the periodic part $\overline{d_{m+1} \dots d_n}$. Thus, we obtain `x=.d1 d2` and `wt=z-x`, which is in this case `x=[-.2;.4]` and `wt=[0;0]`. The option `'check'` ensures that the value of `x` is computed, for given $\texttt{z} \notin \texttt{KD}$. Be aware that the function `num2ordering` sometimes fails to find a correct value of $\texttt{x} \in \texttt{KD}$, due to the fractal nature of the attractors, limited computing time or for points at the boundary of `KD`, and thus the returned value of `x` should be counter-checked by hand. With `T=transitionmatrix(S,'Omega',Om)` we compute the transitionmatrices, which are given by

$$\texttt{T\{1\}} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0
\end{bmatrix}, \quad \texttt{T\{2\}} = \begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&2&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}.$$

The matrix `T{1}T{2}` has a simple, leading eigenvalue 1 with normalized corresponding eigenvector

$$\texttt{v=[0 0 0 0 .1 0 0 .2 .4 0 0 .2 .1 0 0 0]'} \in \mathbb{R}^{\texttt{OmC}}$$

such that `sum(v)=1`. Therefore, since `wt=[0;0]` is the thirteenth entry in `OmC`, $\phi(\texttt{z}) = \texttt{.1}$. The basic limit function can be plotted using `blf(S)`.

The computed sets and points are plotted in Figure 3.6. For convenience of the reader, the raw source code is given in Figure 3.7. △

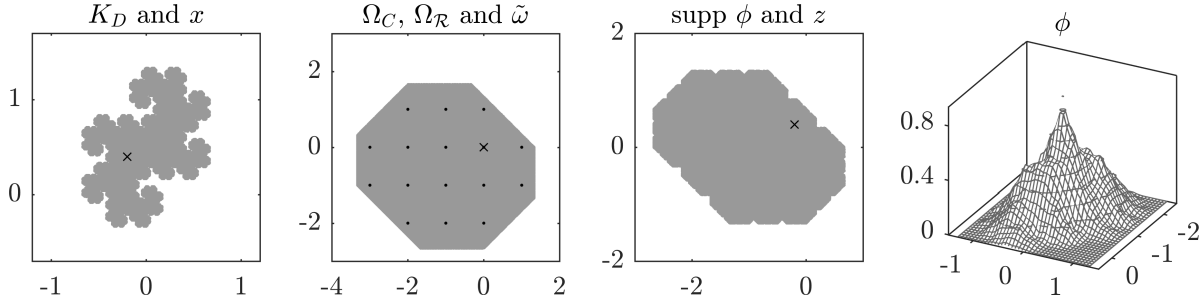


Figure 3.6: Plots for the the stationary subdivision scheme defined by $S = (\frac{1}{2}\begin{bmatrix}1 & 2 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix})$ from Example 3.3.6.

```
a = 1/2*[ 1 2 1 ];                          %define the mask
M = [ 1 1; -1 1 ];                          %define the dilation matrix
S = getS({ a, M })                          %define S
D = S{3}                                    %display digit set
OmC = constructOmega(S)                     %compute Omega_C
T = transitionmatrix(S,'Omega',OmC)         %compute transition matrices
2*T{1}, 2*T{2}                              %display transition matrices
z = [ -.2; .4 ]                             %choose value for z
[ oo,~, x ] = num2ordering(S,z,'check') %compute x and its digit expansion
wt = z-x                                    %get value of omega_tilde
[ v, d ] = eig(T{1}*T{2});                  %compute eigenvectors
d = diag(d)                                 %get index of eigenvalue 1
idx = find(d==1);                           %get eigenvector to 1
v = v/sum(v)                                %normalize v
idxom = find(ismember(OmC',wt','rows')) %get index of omega_tilde
v(idxom)                                    %display value phi(z)

% plot commands
plotm(OmC,'.');                             %plot Om_C
OmRR = tile(getS(S,'OmegaRR'));             %plot Om_RR
KA = tile(getS(S,'supp'));                  %plot K_{supp a}
KD = tile(getS(S));                         %plot K_D
blf(S);                                     %plot phi
```

Figure 3.7: Listing for Example 3.3.6.

3.4 Unifying the matrix and operator approach

In this section we unify the *matrix (joint spectral radius)* and *operator (restricted spectral radius) approach* in the setting of multiple subdivision schemes, Theorem 3.4.17. Theorem 3.4.17 generalizes similar results in [Charina, Conti and Sauer, 2005; Charina, 2012] that were proven in the stationary setting for the case of the dilation matrix $M = 2I$. Thus, our generalization has two directions: first we allow for general dilation matrices, second we allow for multiple subdivision schemes.

One of the standard tools for checking the regularity of subdivision schemes is the so-called *restricted spectral radius* (RSR) that measures the spectral properties of the difference subdivision operators restricted to a certain subspace of $\ell(\mathbb{Z}^s)$, see e.g. [Cavaretta, Dahmen and Micchelli, 1991; Charina, Conti and Sauer, 2005; Charina, 2012; Sauer, 2012]. The concept of the restricted spectral radius relies on the backward difference operators and difference subdivision schemes operating on the sequences in $\ell(\mathbb{Z}^s)$.

Definition 3.4.1. We define

(*i*) the l^{th} *backward difference operator* $\nabla_l : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$ by $\nabla_l c = c - c(\cdot - e_l)$, where e_l, $l \in \{1, \dots, s\}$, are the standard unit vectors of $\mathbb{R}^s$,

(*ii*) the *backward difference operator* $\nabla : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s, \mathbb{R}^s)$ by $\nabla = \begin{bmatrix} \nabla_1 & \nabla_2 & \dots & \nabla_s \end{bmatrix}^T$,

(*iii*) the μ^{th} *backward difference operator* by $\tilde{\nabla}^\mu = \prod_{l=1}^s (\nabla_l)^{\mu_l}$, $\mu \in \mathbb{N}_0^s$ and

(*iv*) the $(k+1)^{st}$ *backward difference operator* ∇^{k+1} by the column vector whose entries are all possible μ^{th} backward differences with $|\mu| = k+1$, $k \in \mathbb{N}_0$.

We illustrate Definition 3.4.1 on the following example.

Example 3.4.2. For dimension $s = 2$ we get

$$\nabla^2\delta = \begin{bmatrix} \tilde{\nabla}^{(2,0)} \\ \tilde{\nabla}^{(1,1)} \\ \tilde{\nabla}^{(0,2)} \end{bmatrix} \delta = \begin{bmatrix} \nabla_1\nabla_1 \\ \nabla_1\nabla_2 \\ \nabla_2\nabla_2 \end{bmatrix} \begin{bmatrix} \mathbf{1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{1} \\ -2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} \mathbf{1} & -1 \\ -1 & 1 \end{bmatrix} \\ \begin{bmatrix} \mathbf{1} & -2 & 1 \end{bmatrix} \end{bmatrix}.$$

Note that the entries in the matrix are bi-infinite sequences and we only list their non-zero elements. △

Lemma 3.4.3. *If $k \in \mathbb{N}_0$, then $\nabla^{k+1} : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s, \mathbb{R}^n)$, $n = \binom{s+k}{s-1}$.*

Proof. To compute the cardinality of the set $\#\{\mu \in \mathbb{N}_0^s : |\mu| = k+1\}$, we must choose $k+1$ positions out of a set of s elements and each position can be chosen multiple times. This is a problem of counting the number of combinations with repetition. Hence the number of choices equals $\binom{s+k}{s-1}$. □

We will often use the following straight forward property of the backward difference operator.

Lemma 3.4.4. *For any $c \in \ell(\mathbb{Z}^s)$, $\mu \in \mathbb{N}_0^s$,*

$$\tilde{\nabla}^\mu c = \tilde{\nabla}^\mu \left(\sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\delta(\cdot - \alpha) \right) = \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\tilde{\nabla}^\mu \, \delta(\cdot - \alpha). \tag{3.4.1}$$

Remark 3.4.5. The function `diffsequence` [Mejstrik, 2018b] implements the backward difference operators ∇^k and $\tilde{\nabla}^\mu$ defined in 3.4.1.

Definition 3.4.6. Let $k \in \mathbb{N}_0$ and $\mathcal{S}$ be a finite set of subdivision operators whose masks satisfy sum rules of order $k+1$.

(*i*) We define a *difference subdivision operator (of order $k+1$)*, $S' = (a', M) : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s,\ \mathbb{R}^n)$, $a' \in \ell_0(\mathbb{Z}^s,\ \mathbb{R}^{n\times n})$, $n = \binom{s+k}{s-1}$, by

$$\nabla^{k+1} S = S' \nabla^{k+1}.$$

(*ii*) By $\mathcal{S}'$ we denote a set of the difference subdivision operators (of order $k+1$) S' associated to the set $\mathcal{S}$ of subdivision operators.

(*iii*) A sequence $(S'_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}'^{\mathbb{N}}$, $j_n \in \{1, \dots, J\}$, is called a *(multiple) difference subdivision scheme* (of order $k+1$).

The existence of difference subdivision operators is ensured by the assumption that the masks of the subdivision operators fulfil sum rules of a certain order, see e.g. [Cavaretta, Dahmen and Micchelli, 1991; Möller and Sauer, 2004; Jia, 1998; Sauer, 2002b].

Definition 3.4.7. For $k \in \mathbb{N}_0$ we define the linear space of *polynomial sequences* of *degree k* by

$$\Pi_k = \left\{ c \in \ell(\mathbb{Z}^s) : c(\alpha) = \sum_{\nu \in \mathbb{N}_0,\ |\nu| \le k} c_\nu \alpha^\nu,\ c_\nu \in \mathbb{R} \right\}.$$

Example 3.4.8. Constant sequences are polynomial sequences of degree 0. The sequence

$$c = \begin{bmatrix} \ddots & & & & \vdots & & & & \iddots \\ & 8 & 3 & 0 & -1 & 0 & 3 & 8 & \\ \cdots & 9 & 4 & 1 & \mathbf{0} & 1 & 4 & 9 & \cdots \\ & 10 & 5 & 2 & 1 & 2 & 5 & 10 & \\ \iddots & & & & \vdots & & & & \ddots \end{bmatrix},$$

given by $c(\alpha) = \alpha_1 + \alpha_2^2$, $\alpha \in \mathbb{Z}^s$, is a polynomial sequence of degree 2. △

Lemma 3.4.9. *Let $k \in \mathbb{N}_0$ and S a subdivision operator. The following statements are equivalent:*

(i) S is a subdivision operator whose mask satisfies sum rules of order $k+1$.

(ii) Π_k is invariant under the subdivision operator S.

(iii) There exists a difference subdivision operator S' of order $k+1$ corresponding to S .

See [Jia, 1998, Theorem 5.2] and [Sauer, 2002b, Theorem 3] for more details.

Note that the space of polynomial sequences Π_k, $k \in \mathbb{N}_0$ is invariant under all subdivision operators whose masks satisfy sum rules of order $k+1$; in particular for subdivision operators corresponding to non-convergent subdivision schemes.

Example 3.4.10. We consider the non-convergent stationary subdivision scheme given by the operator $S = (a, M)$ with the mask $a = \begin{bmatrix} 1 & \frac{1}{2} & \mathbf{-1} & \frac{1}{2} & 1 \end{bmatrix}^T$ and the dilation matrix $M = 2$. By [Daubechies and Lagarias, 1992b, Theorem 2.2], the corresponding subdivision scheme is not convergent. The mask fulfils sum rules of order 2. Indeed, $1 = a_{-2} + a_0 + a_2 = a_{-1} + a_1 = 1$ and $0 = -2a_{-2} + 0a_0 + 2a_2 = -1a_{-1} + 1a_1 = 0$.

Thus, the space Π_1 of polynomials up to degree 1 is invariant under S. To test this, we choose the starting sequence $c(\alpha) = \alpha$ if $\alpha \geq 0$ and 0 otherwise. Due to the locality of the subdivision operator, the refined sequences $S^n c$, $n \in \mathbb{N}$, restricted to large negative and positive indices, should be polynomial sequences; indeed, see Figure 3.8. △

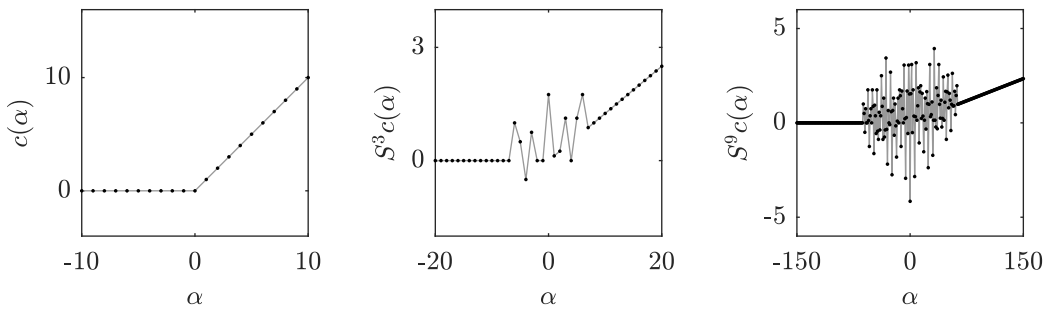


Figure 3.8: The non-convergent subdivision scheme $(S)_{n\in\mathbb{N}}$ from Example 3.4.10 applied to the sequence c, which is a polynomial sequence when restricted to large positive or negative indices.

Difference subdivision schemes, as defined in 3.4.6, are not uniquely determined. Furthermore, contrary to the case of dilation $M = 2I$, for general dilation matrices the support of the masks of the difference schemes can be larger than the support of the original scheme [Sauer, 2002b, Remark 3.7].

Example 3.4.11. Let $S = (a, M)$ with $a = \frac{1}{2}\begin{bmatrix}\mathbf{1} & 2 & 1\end{bmatrix}$ and $M = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$.

The scheme fulfils sum rules of order 2. Indeed, Equations (3.1.4) and (3.1.5) are fulfilled for $\nu \in \{(0,0),\ (1,0),\ (0,1)\}$, $\alpha \in \left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$,

$$1 = \frac{1}{2} + \frac{1}{2} = \sum_{\beta\in\mathbb{Z}^s} a(M\beta + [\,{}^0_0\,]) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta + [\,{}^0_1\,]) = \frac{2}{2} = 1,$$

$$0 = 0\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = \sum_{\beta\in\mathbb{Z}^s} (M\beta + [\,{}^0_0\,])^{(1,0)} a(M\beta + [\,{}^0_0\,]) = \sum_{\beta\in\mathbb{Z}^s} (M\beta + [\,{}^0_1\,])^{(1,0)} a(M\beta + [\,{}^0_1\,]) = 0\cdot\frac{2}{2} = 0,$$

$$1 = 0\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = \sum_{\beta\in\mathbb{Z}^s} (M\beta + [\,{}^0_0\,])^{(0,1)} a(M\beta + [\,{}^0_0\,]) = \sum_{\beta\in\mathbb{Z}^s} (M\beta + [\,{}^0_1\,])^{(0,1)} a(M\beta + [\,{}^0_1\,]) = 1\cdot\frac{2}{2} = 1.$$

By Lemma 3.4.9, there exists a difference scheme S' of order one such that $\nabla^1 S = S'\nabla^1$. We show next, there exists no difference scheme with a mask

$$a' = \begin{bmatrix} a^{(1,1)'} & a^{(1,2)'} \\ a^{(2,1)'} & a^{(2,2)'} \end{bmatrix},$$

such that $\operatorname{supp} a^{(i,j)'} \subseteq \operatorname{supp} a$, $i, j \in \{1, 2\}$. Assume the contrary. To determine the sequences $a^{(i,j)'}$, we solve the linear system $\nabla^1 S\delta = S'\nabla^1\delta$. We first compute

$$\nabla S\delta = \nabla a = \frac{1}{2}\begin{bmatrix}\begin{bmatrix}\mathbf{1} & 2 & 1\\ -1 & -2 & -1\end{bmatrix} \\ \begin{bmatrix}\mathbf{1} & 1 & -1 & -1\end{bmatrix}\end{bmatrix}$$

and

$$\begin{aligned} S'\nabla\delta &= \sum_{\beta\in\mathbb{Z}^s} \begin{bmatrix} a^{(1,1)'} & a^{(1,2)'} \\ a^{(2,1)'} & a^{(2,2)'} \end{bmatrix} (\,\cdot - M\beta) \begin{bmatrix}\begin{bmatrix}\mathbf{1}\\ -1\end{bmatrix} \\ \begin{bmatrix}\mathbf{1} & -1\end{bmatrix}\end{bmatrix}(\beta) \\ &= \begin{bmatrix} a^{(1,1)'} + a^{(1,2)'} - a^{(1,1)'}(\,\cdot - \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]) - a^{(1,2)'}(\,\cdot - [\,{}^1_1\,]) \\ a^{(2,1)'} + a^{(2,2)'} - a^{(2,1)'}(\,\cdot - \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]) - a^{(2,2)'}(\,\cdot - [\,{}^1_1\,]) \end{bmatrix}. \end{aligned}$$

Again, note that the entries in the matrices are bi-infinite sequences and we only list the non-zero entries. Thus, we would have to solve the following system of equations:

$$\begin{bmatrix} 0 & a_{0,1}^{(1,1)'} + a_{0,1}^{(1,2)'} & a_{0,2}^{(1,1)'} + a_{0,2}^{(1,2)'} & a_{0,3}^{(1,1)'} + a_{0,3}^{(1,2)'} & 0 \\ -a_{0,1}^{(1,1)'} & -a_{0,2}^{(1,1)'} & -a_{0,3}^{(1,1)'} - a_{0,1}^{(1,2)'} & -a_{0,2}^{(1,2)'} & -a_{0,3}^{(1,2)'} \end{bmatrix} = \\ = \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & a_{0,1}^{(2,1)'} + a_{0,1}^{(2,2)'} & a_{0,2}^{(2,1)'} + a_{0,2}^{(2,2)'} & a_{0,3}^{(2,1)'} + a_{0,3}^{(2,2)'} & \underline{0} \\ -a_{0,1}^{(2,1)'} & -a_{0,2}^{(2,1)'} & -a_{0,3}^{(2,1)'} - a_{0,1}^{(2,2)'} & -a_{0,2}^{(2,2)'} & -a_{0,3}^{(2,2)'} \end{bmatrix} = \\ = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & -1 & \underline{-1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As one can see from the two underlined numbers $\underline{0}$ and $\underline{-1}$, this system has no solution. A possible solution with a mask with larger support is given by

$$a^{(1,1)'} = \frac{1}{2} \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 \end{bmatrix}, \qquad a^{(1,2)'} = \frac{1}{2} \begin{bmatrix} \mathbf{0} & 1 & 1 & 0 \end{bmatrix},$$
$$a^{(2,1)'} = \frac{1}{2} \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 \end{bmatrix}, \qquad a^{(2,2)'} = \frac{1}{2} \begin{bmatrix} \mathbf{0} & 0 & -1 & -1 \end{bmatrix}.$$

Another solution is given by

$$a^{(1,1)'} = \frac{1}{4} \begin{bmatrix} 0 & -1 & -1 & 0 \\ \mathbf{0} & 0 & -1 & -1 \end{bmatrix}, \qquad a^{(1,2)'} = \frac{1}{4} \begin{bmatrix} 0 & 1 & 1 & 0 \\ \mathbf{1} & 1 & 0 & 0 \end{bmatrix},$$
$$a^{(2,1)'} = \frac{1}{2} \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 \end{bmatrix}, \qquad a^{(2,2)'} = \frac{1}{2} \begin{bmatrix} \mathbf{0} & 0 & -1 & -1 \end{bmatrix}. \qquad \triangle$$

In the setting of multiple subdivision, we use the following definition of the restricted spectral radius.

Definition 3.4.12 ([Sauer, 2012, Section 3, "*normalized joint spectral radius*"])**.** Let $k \in \mathbb{N}_0$ and $\mathcal{S}$ be a finite set of subdivision operators whose masks satisfy sum rules of order $k+1$ and $\mathcal{S}'$ an associated set of difference subdivision operators of order $k+1$. The $(k+1)^{st}$ *restricted norm* of $S' \in \mathcal{S}'$ is defined by

$$\left|\left| S'|_{\nabla^{k+1}} \right|\right|_\infty = \sup_{\substack{c \in \ell_\infty(\mathbb{Z}^s) \\ ||\nabla^{k+1} c||_\infty = 1}} \left|\left| S' \nabla^{k+1} c \right|\right|_\infty . \tag{3.4.2}$$

The $(k+1)^{st}$ *restricted spectral radius* (RSR) of $\mathcal{S}'$ is defined by

$$\mathrm{RSR}(\mathcal{S}'|_{\nabla^{k+1}}) = \limsup_{n \to \infty} \sup_{S'_j \in \mathcal{S}'} \left|\left| S'_{j_n} \cdots S'_{j_1} |_{\nabla^{k+1}} \right|\right|_\infty^{1/n}.$$

Difference schemes are used to characterize the convergence properties of multiple subdivision schemes.

Theorem 3.4.13. *[Sauer, 2012, Theorem 2]* $\mathcal{S}^{\mathbb{N}}$ *is convergent if and only if there exists a corresponding set* $\mathcal{S}'$ *of difference subdivision operators of order 1, such that* $\mathrm{RSR}(\mathcal{S}'|_\nabla) < 1$.

The main result of this section, Theorem 3.4.17, leads to a characterization of convergence of $\mathcal{S}^{\mathbb{N}}$ in terms of the joint spectral radius of the transition matrices restricted to a common invariant subspace. The key to understanding is Lemma 3.4.15, which describes the invariant subspaces for the transition operators in the matrix approach. These are the analogon to the invariant subspace of polynomial sequences for the subdivision operators in the operator approach.

Definition 3.4.14. Let $\Omega \subseteq \mathbb{Z}^s$ be finite and $k \in \mathbb{N}_0$. We define the linear spaces

(*i*) $V_k(\Omega) = \left\{ v \in \mathbb{R}^\Omega : \sum_{\beta\in\Omega} v(\beta)p(-\beta) = 0 \text{ for all } p \in \Pi_k \right\}$ and

(*ii*) $\tilde{V}_k(\Omega) = \operatorname{span}\left\{ v \in \ell_0(\mathbb{Z}^s) : v = \tilde{\nabla}^\mu \delta(\cdot - \beta),\ \beta \in \mathbb{Z}^s,\ \mu \in \mathbb{N}_0^s,\ |\mu| = k+1,\ \operatorname{supp} v \subseteq \Omega \right\}$.

Note that we identify finitely supported sequences with vectors, matrices or tensors.

Lemma 3.4.15. *If $\Omega \subseteq \mathbb{Z}^s$ is finite and $k \in \mathbb{N}_0$, then*

(*i*) $\tilde{V}_k(\Omega) \subseteq V_k(\Omega)$ *and, if* $\dim V_k(\Omega) = \dim \tilde{V}_k(\Omega)$, *then* $V_k(\Omega) = \tilde{V}_k(\Omega)$.

(*ii*) $V_{k+1}(\Omega) \subseteq V_k(\Omega)$.

(*iii*) $\tilde{V}_{k+1}(\Omega) \subseteq \tilde{V}_k(\Omega)$.

Proof. (*i*) Let $p \in \Pi_k$. If $v \in \tilde{V}_k(\Omega)$, then there exist $v_{\mu,\beta} \in \mathbb{R}^s$ such that

$$v = \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_{\mu,\beta} \tilde{\nabla}^\mu \delta(\cdot - \beta).$$

Using Lemma 3.4.4 and the fact that $\tilde{\nabla}^\mu p = 0$ for all $|\mu| \geq k+1$ we obtain.

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}^s} v(\alpha)p(-\alpha) &= \sum_{\alpha\in\mathbb{Z}^s} \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_{\mu,\beta} \tilde{\nabla}^\mu \delta(\alpha-\beta)p(-\alpha) \\
&= \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_{\mu,\beta} \sum_{\alpha\in\mathbb{Z}^s} \tilde{\nabla}^\mu \delta(-\beta+\alpha)p(-\alpha) \\
&= \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_{\mu,\beta} \tilde{\nabla}^\mu \left(\sum_{\alpha\in\mathbb{Z}^s} \delta(\cdot+\alpha)p(-\alpha) \right)(-\beta) \\
&= \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_{\mu,\beta} \tilde{\nabla}^\mu p(-\beta) = 0.
\end{aligned}$$

By definition of $\tilde{V}_k(\Omega)$, $\operatorname{supp} v \subseteq \Omega$, and thus $\tilde{V}_k(\Omega) \subseteq V_k(\Omega)$.

(*ii*) Let $k \in \mathbb{N}$, $v \in V_{k+1}(\Omega)$. By Definition 3.4.14 of V_k, $\sum_{\beta\in\mathbb{Z}^s} v(\beta)p(-\beta) = 0$ for all $p \in \Pi_{k+1}$. In particular, since $\Pi_k \subseteq \Pi_{k+1}$, $\sum_{\beta\in\mathbb{Z}^s} v(\beta)p(-\beta) = 0$ for all $p \in \Pi_k$, which implies that $v \in V_k(\Omega)$.

(*iii*) Let $v = \tilde{\nabla}^\mu \delta(\cdot - \beta) \in \ell(\Omega)$ with $\beta \in \mathbb{Z}^s$, $\mu \in \mathbb{N}_0^s$, $|\mu| = k+2$. Since $\mu = (\mu_1, \ldots, \mu_s) \neq 0$, there exists $l \in \{1, \ldots, s\}$ such that $\mu_l > 0$. By Definition 3.4.1, $\tilde{\nabla}^\mu \delta(\cdot - \beta) = \tilde{\nabla}^{\mu-e_l}\delta(\cdot - \beta) - \tilde{\nabla}^{\mu-e_l}\delta(\cdot - \beta - e_l)$, $|\mu - e_l| = k+1$ and $\operatorname{supp} \tilde{\nabla}^{\mu-e_l}\delta(\cdot - \beta) \subseteq \Omega$, $\operatorname{supp} \tilde{\nabla}^{\mu-e_l}\delta(\cdot - \beta - e_l) \subseteq \Omega$. Therefore, $v \in \tilde{V}_k(\Omega)$. □

The existence of difference subdivision operators S' translates in the setting of the matrix approach and becomes the invariance of the spaces $V_k(\Omega)$, defined in 3.4.14, under the transition operator $\mathcal{T}_d \in \mathcal{T}$. Thus, Lemma 3.4.16 can be seen as an analogon of Lemma 3.4.9.

Lemma 3.4.16 ([Cavaretta, Dahmen and Micchelli, 1991, Section 3.5], [Jia, 1998, Theorem 5.2], [Charina, 2012, Lemma 4.4])**.** *Let $k \in \mathbb{N}_0$. If $S = (a, M)$ is a subdivision operator whose mask a satisfies sum rules of order $k+1$ and $\Omega \subseteq \mathbb{Z}^s$ such that $\ell(\Omega)$ is $\mathcal{T}$-invariant for all $d \in D$, then $V_n(\Omega)$ is $\mathcal{T}$-invariant for all $n \leq k$.*

Proof. Let $n \le k$, $k \in \mathbb{N}$, and $v \in V_n(\Omega)$. If $p \in \Pi_n$ and $d \in D$, then, by (3.2.1),

$$\sum_{\alpha\in\mathbb{Z}^s} T_d v(\alpha)p(-\alpha) = \sum_{\beta\in\Omega} v(\beta) \sum_{\alpha\in\mathbb{Z}^s} a(M\alpha-\beta+d)p(-\alpha) = \sum_{\beta\in\Omega} v(\beta)(Sp)(d-\beta).$$

By Lemma 3.4.9 (*ii*), $(Sp)(d-\,\cdot\,) \in \Pi_n$, and thus $\sum_{\alpha\in\mathbb{Z}^s} T_d v(\alpha)p(-\alpha) = 0$ which implies that $T_{d,\Omega} v \in V_n(\Omega)$. □

We are now ready to formulate the main result of this section,

Theorem 3.4.17. *Let $k \in \mathbb{N}_0$ and $\mathcal{S}$ be a finite set of subdivision operators whose masks satisfy sum rules of order $k+1$. If there exists a finite set $\Omega \subseteq \mathbb{Z}^s$ such that*

(*i*) *$\ell(\Omega)$ is invariant under the transition operators in $\mathcal{T}$,*

(*ii*) *$V_k(\Omega) = \tilde{V}_k(\Omega)$ and*

(*iii*) *$X_\mu \neq \emptyset$ for all $|\mu| = k+1$, where $X_\mu = \left\{\beta \in \mathbb{Z}^s : \operatorname{supp} \tilde{\nabla}^\mu \delta(\,\cdot\,-\beta) \subseteq \Omega\right\}$,*

then $\operatorname{RSR}(\mathcal{S}'|_{\nabla^{k+1}}) = \operatorname{JSR}(\{T_{d,j,\Omega}|_{V_k(\Omega)} : d \in D_j,\ j = 1,\dots,J\})$.

We give the proof of Theorem 3.4.17 in Section 3.5.

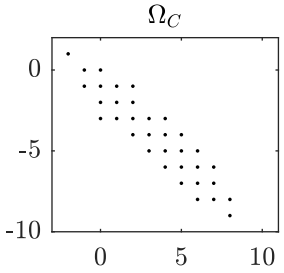
Remark 3.4.18. Example 3.5.5 shows that assumption (*ii*) and (*iii*) of Theorem 3.4.17 are indeed crucial. The natural candidate for the set Ω in Theorem 3.4.17 would be the set Ω_C from Lemma 3.2.9. The set Ω_C, by Lemma 3.2.9, satisfies assumption (*i*) of Theorem 3.4.17 and our numerical experiments show that in most cases Ω_C also satisfies the assumptions (*ii*) and (*iii*). However, Example 3.4.19 illustrates that the case $\tilde{V}_0(\Omega_C) \subsetneq V_0(\Omega_C)$ and $X_\mu = \emptyset$, for some $\mu \in \mathbb{N}_0^s$, occurs sometimes even in the stationary setting. In such cases, we choose $\Omega = \Omega_V$ from Proposition 3.4.21 and study, if necessary, the convergence properties of an iterated subdivision scheme $\mathcal{S}^n$, $n \in \mathbb{N}$, whose dilation matrices M_j satisfy $\left\|M_j^{-1}\right\|_\infty < 1$.

The structure of Ω_V is adapted to the definition of the restricted spectral radius and makes the link between the two spectral radii more evident. The definition of the set Ω_V is straightforward in comparison to the set Ω_C from Lemma 3.2.9. However, the latter is by far more efficient for numerical computations.

Example 3.4.19. Consider the dilation matrix $M = \begin{bmatrix} -3 & -4 \\ 4 & 4 \end{bmatrix}$ with digit set $D = \left\{ \begin{bmatrix} -m \\ m \end{bmatrix} : m = 0,1,2,3 \right\}$ and choose any mask a with support $\operatorname{supp} a = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$, for example

$$a = \frac{1}{6} \begin{bmatrix} \mathbf{0} & 0 & 0 & 6 & 0 \\ 2 & 0 & 6 & 0 & 2 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{bmatrix},$$

which, by [Daubechies and Lagarias, 1992b, Theorem 2.2], corresponds to a convergent subdivision scheme whose basic limit function has Hölder regularity $\alpha \simeq 0.1200$. The set Ω_C constructed by Algorithm 3.2.9 is plotted in Figure 3.9. Straightforward computation shows that $\dim V_0(\Omega_C) = 33 > \dim \tilde{V}_0(\Omega_C) = 32$. Thus, Ω_C is inappropriate for further theoretical analysis. The problematic point is $\begin{bmatrix} -2 & 1 \end{bmatrix}^T$ which has no direct neighbour. The dimensions of the spaces $V_k(\Omega_C)$ and $\tilde{V}_k(\Omega_C)$, $k \in \{0,\dots,10\}$, are also printed in Figure 3.9. △



k	0	1	2	3	4	5	6	7	8	9	10
$\dim V_k(\Omega_C)$	33	31	28	24	19	15	11	7	4	1	0
$\dim \tilde{V}_k(\Omega_C)$	32	29	22	2	0	0	0	0	0	0	0

Figure 3.9: The set Ω_C and the dimensions of the spaces $V_k(\Omega_C)$ and $\tilde{V}_k(\Omega_C)$ from Example 3.4.19.

In Lemma 3.4.20 we introduce a family of finite sets $\Omega_V \subseteq \mathbb{Z}^s$, such that $V_k(\Omega_V) = \tilde{V}_k(\Omega_V)$ for all $k \in \mathbb{N}_0$.

Lemma 3.4.20. *If* $\Omega = ([0, N_1] \times \cdots \times [0, N_s]) \cap \mathbb{Z}^s$, $N_l \in \mathbb{N}_0$, $l \in \{1, \ldots, s\}$, *then* $V_k(\Omega) = \tilde{V}_k(\Omega)$ *for all* $k \in \mathbb{N}_0$.

Proof. The proof is by induction on k. The proof of the induction step is based on [Charina, 2012, Proof of Lemma 4.4].

Let $k = 0$. The dimension of $V_0(\Omega)$ is $\#\Omega - 1$, due to $V_0(\Omega)$ being orthogonal to the vector of all ones. To determine the dimension of $\tilde{V}_0(\Omega)$, we consider the graph $G = (\Omega, E)$ with the set of edges

$$E = \left\{(\omega_1, \omega_2) \in \Omega^2 : ||\omega_1 - \omega_2||_1 = 1\right\}.$$

Using this point of view, every sequence of the form $\nabla_l \, \delta(\cdot - \beta) \in \tilde{V}_0(\Omega)$, $\beta \in \mathbb{Z}^s$, $l \in \{1, \ldots, s\}$, is associated uniquely to an edge in E. The graph G is connected, thus, there exists a corresponding spanning tree consisting of $\#\Omega - 1$ edges from E [Diestel, 2005, Theorem 1.5.1]. Since any spanning tree does not contain cycles, the set of edges of the spanning tree corresponds to a set of linearly independent sequences in $\tilde{V}_0(\Omega)$. Thus, $\dim \tilde{V}_0(\Omega) = \#\Omega - 1$.

Assume the statement holds for all natural numbers less than k, $k \geq 1$. We show next that the statement holds for k. Let $v \in V_k(\Omega)$, by Lemma 3.4.15, $v \in V_{k-1}(\Omega)$. By the induction hypotheses $v \in \tilde{V}_{k-1}(\Omega)$, and thus is of the form

$$v = \sum_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k}} \sum_{\beta \in X_\mu} v_\mu(\beta) \tilde{\nabla}^\mu \, \delta(\cdot - \beta), \tag{3.4.3}$$

where $X_\mu = \left\{\beta \in \mathbb{Z}^s : \operatorname{supp} \tilde{\nabla}^\mu \, \delta(\cdot - \beta) \subseteq \Omega\right\} = ([0, N_1 - \mu_1] \times \cdots \times [0, N_s - \mu_s]) \cap \mathbb{Z}^s$, $N_l > \mu_l$ and $v_\mu(\beta) \in \mathbb{R}$; if $N_l < \mu_l$ for some $l \in \{1, \ldots, s\}$, then $X_\mu = \emptyset$. By the definition of $V_k(\Omega)$ and (3.4.3), for all $p \in \Pi_k$,

$$0 = \sum_{\alpha \in \mathbb{Z}^s} v(\alpha) p(-\alpha) = \sum_{\alpha \in \mathbb{Z}^s} \sum_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k}} \sum_{\beta \in X_\mu} v_\mu(\beta) \tilde{\nabla}^\mu \, \delta(\alpha - \beta) p(-\alpha) = \sum_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k}} \sum_{\beta \in X_\mu} v_\mu(\beta) \tilde{\nabla}^\mu \, p(-\beta).$$

If we choose $p(\alpha) = \alpha^\nu$, $|\nu| = k$, then $\tilde{\nabla}^\mu \, p(-\beta) = 0$ whenever $\nu \neq \mu$ and constant otherwise, which implies

$$\sum_{\beta \in X_\mu} v_\mu(\beta) = 0 \quad \text{for all} \quad |\mu| = k.$$

If we identify the sequences v_μ with the Laurent polynomials $v_\mu^*(z) = \sum_{\beta\in X_\mu} v_\mu(\beta)z^\beta$, $z \in \mathbb{C}^s \setminus \{0\}$, we see that all v_μ^* vanish at $z = (1,\dots,1) \in \mathbb{C}^s$. Because $\{1-z_1,\dots,1-z_s\}$ is a Gröbner basis with respect to the lexicographic ordering of monomials (see e.g. [Cox, Little and O'Shea, 2015],[Sauer, 2002a]), there exist polynomials $v_{\mu,l}^*$ such that $\operatorname{supp} v_{\mu,l} \subseteq X_{\mu,l}$, $X_{\mu,l} = \big([0,N_1-\mu_1]\times\cdots[0,N_l-\mu_l-1]\times\cdots\times[0,N_s-\mu_s]\big)\cap\mathbb{Z}^s$ and

$$v_\mu^*(z) = \sum_{l=1}^s (1-z_l)v_{\mu,l}^*(z) = \sum_{l=1}^s \left(\nabla_l\, v_{\mu,l}\right)^*(z).$$

The polynomials $v_{\mu,l}^*$ can be computed by polynomial long division for example. Thus, we get

$$v_\mu(\beta) = \sum_{l=1}^s \sum_{\gamma\in X_{\mu,l}} v_{\mu,l}(\gamma)\nabla_l\,\delta(\beta-\gamma). \tag{3.4.4}$$

Therefore, since $|\mu+e_l| = k+1$, $l\in\{1,\dots,s\}$, (3.4.3) and (3.4.4) yield

$$v = \sum_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k}}\sum_{\beta\in X_{\mu,l}}\sum_{l=1}^s \nabla_l\, v_{\mu,l}(\beta)\tilde\nabla^\mu\,\delta(\cdot-\beta) = \sum_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k}}\sum_{\beta\in X_{\mu,l}}\sum_{l=1}^s v_{\mu,l}(\beta)\tilde\nabla^{\mu+e_l}\delta(\cdot-\beta),$$

showing that $v\in\tilde V_k(\Omega)$.

The inclusion $\tilde V_k(\Omega)\subseteq V_k(\Omega)$, $k\in\mathbb{N}_0$, follows by Lemma 3.4.15 (i). □

Proposition 3.4.21. *Let $\mathcal{S}$ be a finite set of subdivision operators whose dilation matrices M_j, $j\in\{1,\dots,J\}$, satisfy*

$$\left|\left|M_j^{-1}\right|\right|_\infty < 1. \tag{3.4.5}$$

For the set $\mathcal{T}$ of transition operators, there exists a finite set $\Omega_V\subseteq\mathbb{Z}^s$ such that

(i) $\ell(\Omega_V)$ is invariant under operators in $\mathcal{T}$,

(ii) $V_k(\Omega_V) = \tilde V_k(\Omega_V)$ for all $k\in\mathbb{N}_0$ and

(iii) $X_\mu\neq\emptyset$ for all $|\mu| = k+1$, where $X_\mu = \left\{\beta\in\mathbb{Z}^s : \operatorname{supp}\tilde\nabla^\mu\,\delta(\cdot-\beta)\subseteq\Omega\right\}$.

Proof. By (3.4.5)

$$C_M = \max_{j\in\{1,\dots,J\}}\left|\left|M_j^{-1}\right|\right|_\infty < 1.$$

Due to the finite support of the masks and finiteness of the digit sets, we get constants

$$\begin{aligned} C_a &= \max\left\{||\alpha||_\infty : \alpha\in\operatorname{supp} a_j,\ j=1,\dots,J\right\} < \infty \quad\text{and}\\ C_D &= \max\left\{||d||_\infty : d\in D_j,\ j=1,\dots,J\right\} < \infty. \end{aligned}$$

We define

$$\Omega_V = \left\{x\in\mathbb{R}^s : ||x||_\infty \le \max\left\{\frac{C_a+C_D}{1-C_M}, k+2\right\}\right\}\cap\mathbb{Z}^s.$$

(i) Let $d\in D_j$, $j\in\{1,\dots,J\}$ and $v\in\ell(\Omega_V)$. By (3.2.1), $\mathcal{T}_{d,j}v(\alpha)\neq 0$, whenever $\alpha\in\mathbb{Z}^s$ is such that $M_j\alpha-\beta+d\in\operatorname{supp} a_j$ for some $\beta\in\operatorname{supp} v\subseteq\Omega_V$, or, equivalently, $\alpha\in M_j^{-1}(\operatorname{supp} a_j - d+\Omega_V)$. Thus, we obtain

$$||\alpha||_\infty \le C_M\left(C_a+C_D+\frac{C_a+C_D}{1-C_M}\right) = \frac{C_a+C_D}{1-C_M}(C_M(1-C_M)+C_M) \le \frac{C_a+C_D}{1-C_M},$$

since $-C_M^2+2C_M\le 1$, implying $\mathcal{T}_{d,j}v\in\ell(\Omega_V)$.

(ii) This follows by 3.4.20.

(iii) Since Ω_V contains the set $[-k-2,k+2]^s\cap\mathbb{Z}^s$, it follows that $X_\mu\neq\emptyset$. □

Remark 3.4.22. The proof of Proposition 3.4.21 explains the phenomenon occurring in Example 3.4.19. The graph corresponding to the set Ω_C from this example consists of two connected components. This fact forces $\dim V_0(\Omega_C) > \dim \tilde{V}_0(\Omega_C)$.

Remark 3.4.23. The functions `constructV` and `constructVt` [Mejstrik, 2018b] construct bases for the spaces $V_k(\Omega)$ and $\tilde{V}_k(\Omega)$ respectively. The function `dimVVt` computes the dimensions of the spaces $V_k(\Omega)$ and $\tilde{V}_k(\Omega)$.

3.5 Proof of Theorem 3.4.17

The proof of Theorem 3.4.17 is split into several lemmata, which altogether show that the averaged norm of products of the restricted transition matrices, arising Definition 2.1.1 of the JSR , is equivalent to the restricted norm in (3.4.2).

Lemma 3.5.1 is a generalization of [Charina, 2012, Proposition 4.1]. It shows that, for the computation of the restricted norms of difference subdivision operators in Definition 3.4.12, the supremum over all sequences $c \in \ell_\infty(\mathbb{Z}^s)$ in (3.4.2) is attained by a finitely supported sequence, and thus the supremum can be replaced by a maximum, In particular, this can be used for explicit computation of the restricted norms by means of Linear Programming [Charina, Conti and Sauer, 2005, Paragraph 4].

Lemma 3.5.1. *Let $k \in \mathbb{N}_0$. If $(S_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ is a subdivision scheme and $(S'_{j_n})_{n\in\mathbb{N}} \in \mathcal{S}'^{\mathbb{N}}$ is an associated difference scheme of order $k+1$, i.e.*

$$\nabla^{k+1} S_j = S'_j \nabla^{k+1} \quad \text{for all} \quad j \in \{1, \dots, J\},$$

then there exists a finite set $K \subseteq \mathbb{Z}^s$, depending only on $\mathcal{S}$, such that

$$\left|\left| S'_{j_n} \cdots S'_{j_1}|_{\nabla^{k+1}} \right|\right|_\infty = \max_{\substack{c\in\ell_\infty(K) \\ ||\nabla^{k+1} c||_\infty = 1}} \max_{\alpha \in M_{j_n}\cdots M_{j_1}[0,1)^s \cap \mathbb{Z}^s} \left|\left| S'_{j_n} \cdots S'_{j_1} \nabla^{k+1} c(\alpha) \right|\right|_\infty .$$

Proof. By definition of S'_j and (3.4.1),

$$\begin{aligned}
\left|\left| S'_{j_n} \cdots S'_{j_1}|_{\nabla^{k+1}} \right|\right|_\infty &= \sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s) \\ ||\nabla^{k+1} c||_\infty = 1}} \sup_{\alpha\in\mathbb{Z}^s} \left|\left| S'_{j_n} \cdots S'_{j_1} \nabla^{k+1} c(\alpha) \right|\right|_\infty \\
&= \sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s) \\ ||\nabla^{k+1} c||_\infty = 1}} \sup_{\alpha\in\mathbb{Z}^s} \left|\left| \sum_{\beta\in\mathbb{Z}^s} (S'_{j_n} \cdots S'_{j_1} \delta I)(\alpha - M_{j_n} \cdots M_{j_1}\beta) \nabla^{k+1} c(\beta) \right|\right|_\infty .
\end{aligned}$$

Since there are $\#(M_{j_n} \cdots M_{j_1}[0,1)^s \cap \mathbb{Z}^s)$ different subdivision rules at the n^{th} level of the subdivision recursion, it suffices to take $\alpha \in M_{j_n} \cdots M_{j_1}[0,1)^s \cap \mathbb{Z}^s$. Lemma 3.4.9 (*iii*) in particular shows that the supports of the masks a'_j, $j \in \{1, \dots, J\}$ of the difference schemes S'_j are finite. By Lemma 3.1.13 (*ii*), it follows that

$$\alpha - M_{j_n} \cdots M_{j_1}\beta \in \operatorname{supp}(S'_{j_n} \cdots S'_{j_1}\delta I) \subseteq M_{j_n} \cdots M_{j_2} \operatorname{supp} a'_{j_1} + M_{j_n} \cdots M_{j_3} \operatorname{supp} a'_{j_2} + \cdots + \operatorname{supp} a'_{j_n},$$

and thus, with $(\operatorname{supp} a' \cup \{0\})_j = \operatorname{supp} a'_j \cup \{0\}$,

$$\begin{aligned}
\beta &\in M_{j_1}^{-1} \cdots M_{j_n}^{-1} \big(\alpha - (M_{j_n} \cdots M_{j_2} \operatorname{supp} a'_{j_1} + M_{j_n} \cdots M_{j_3} \operatorname{supp} a'_{j_2} + \cdots + \operatorname{supp} a'_{j_n})\big) \\
&\subseteq M_{j_1}^{-1} \cdots M_{j_n}^{-1} \big((M_{j_n} \cdots M_{j_1}[0,1)^s \cap \mathbb{Z}^s) - (M_{j_n} \cdots M_{j_2} \operatorname{supp} a'_{j_1} + M_{j_n}{}' + \cdots + \operatorname{supp} a'_{j_n})\big) \\
&\subseteq ([0,1)^s \cap M_{j_1}^{-1} \cdots M_{j_n}^{-1}\mathbb{Z}^s) - (M_{j_1}^{-1}(\operatorname{supp} a'_{j_1} \cup \{0\}) + \cdots + M_{j_1}^{-1} \cdots M_{j_n}^{-1}(\operatorname{supp} a'_{j_n} \cup \{0\})) \\
&\subseteq [0,1)^s - K_{\operatorname{supp} a' \cup \{0\}, \boldsymbol{j}},
\end{aligned}$$

where we used that $0 \in \operatorname{supp} a'_j \cup \{0\}$ in the last step. Note that, in view of Example 3.4.11, even if $0 \in \operatorname{supp} a_j$ for all $j \in \{1, \dots, J\}$, we cannot assume without loss of generality that $0 \in \operatorname{supp} a'_j$, $j \in \{1, \dots, J\}$. We denote $K = [0,1)^s - K_{\operatorname{supp} a' \cup \{0\}, \boldsymbol{j}}$. □

Lemma 3.5.2. *If $\mu \in \mathbb{N}_0^s$, $k \in \mathbb{N}_0$, $|\mu| = k$, then*

(i) $\left|\left|\tilde{\nabla}^\mu \delta\right|\right|_\infty = \binom{\mu}{\lfloor \frac{\mu}{2} \rfloor}$ *and*

(ii) $\left|\left|\nabla^k \delta\right|\right|_\infty = \binom{k}{\lfloor \frac{k}{2} \rfloor}$,

where the floor function $\lfloor \cdot \rfloor$ is defined component-wise.

Proof. *(i)* Let $\mu \in \mathbb{N}_0^s$. Straightforward computation shows that

$$\left|\left|\tilde{\nabla}^\mu \delta\right|\right|_\infty = \max_{\nu \le \mu} \left|(-1)^\nu \binom{\mu}{\nu}\right| = \max_{\nu_1 \le \mu_1} \binom{\mu_1}{\nu_1} \cdots \max_{\nu_s \le \mu_s} \binom{\mu_s}{\nu_s} = \binom{\mu_1}{\lfloor \frac{\mu_1}{2} \rfloor} \cdots \binom{\mu_s}{\lfloor \frac{\mu_s}{2} \rfloor} = \binom{\mu}{\lfloor \frac{\mu}{2} \rfloor}.$$

(ii) Let $k \in \mathbb{N}_0$ and define for $\mu \in \mathbb{N}_0^s$, $|\mu| = k$, the function $f_k : \mathbb{N}_0^s \to \mathbb{R}$ by $f_k(\mu) = \binom{\mu}{\lfloor \frac{\mu}{2} \rfloor}$. We need to determine the maximum of f_k. Without loss of generality, due to the definition of the multivariate binomial coefficient (1.2.1), we can assume that $\mu_l \le \lfloor \frac{k}{2} \rfloor$ for all $l \in \{1, \dots, s-1\}$. We first prove that $f_k(\mu) > f_k(\nu)$ for all $\mu, \nu \in \mathbb{N}_0^s$, $|\mu| = |\nu| = k$, $\mu_l \le \nu_l \le \lfloor \frac{k}{2} \rfloor$ for $l \in \{1, \dots, s-1\}$ and $\nu - \mu \in 2\mathbb{N}_0^s$. Loosely speaking, f_k is coordinate-wise monotone for steps of length 2 under reasonable assumptions. If we fix all the coordinates except two, the problem reduces to showing that $X_a / Y_a < 1$ for all $a \in \mathbb{N}_0$ with $2 \le a \le \lfloor \frac{k}{2} \rfloor$, where

$$X_a = \binom{a}{\lfloor \frac{a}{2} \rfloor} \binom{k-a}{\lfloor \frac{k-a}{2} \rfloor}, \quad Y_a = \binom{a-2}{\lfloor \frac{a-2}{2} \rfloor} \binom{k-a+2}{\lfloor \frac{k-a+2}{2} \rfloor}.$$

Using the definition of the binomial coefficient this becomes

$$\begin{aligned}
\frac{X_a}{Y_a} &= \frac{\frac{a!}{\lfloor \frac{a}{2} \rfloor!\ \left(a - \lfloor \frac{a}{2} \rfloor\right)!} \cdot \frac{(k-a)!}{\lfloor \frac{k-a}{2} \rfloor!\ \left(k-a - \lfloor \frac{k-a}{2} \rfloor\right)!}}{\frac{(a-2)!}{\lfloor \frac{a-2}{2} \rfloor!\ \left(a-2 - \lfloor \frac{a-2}{2} \rfloor\right)!} \cdot \frac{(k-a+2)!}{\lfloor \frac{k-a+2}{2} \rfloor!\ \left(k-a+2 - \lfloor \frac{k-a+2}{2} \rfloor\right)!}} \\
&= \frac{a(a-1) \left\lfloor \frac{2+k-a}{2} \right\rfloor \left\lfloor \frac{3+k-a}{2} \right\rfloor}{\lfloor \frac{a}{2} \rfloor \lfloor \frac{a+1}{2} \rfloor (k-a+1)(k-a+2)} \\
&= \frac{4(a-1) \left\lfloor \frac{k-a+2}{2} \right\rfloor \left\lfloor \frac{k-a+3}{2} \right\rfloor}{a(k-a+2)(k-a+1)} \quad \text{for even and odd } k \\
&= \frac{(k-a+2)(a-1)}{a(k-a+1)} \quad \text{for even and odd } a.
\end{aligned}$$

The last term is less then 1 for $2 \le a \le \lfloor \frac{k}{2} \rfloor$. Indeed,

$$\frac{(k-a+2)(a-1)}{a(k-a+1)} - 1$$

is continuous in a, has no poles and zeros on $1 \le a \le k/2$ and is negative at $a = 1$. Now the proof is complete by noting that

$$\binom{1}{\lfloor \frac{1}{2} \rfloor} \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor} \le \binom{k}{\lfloor \frac{k}{2} \rfloor}, \quad k \in \mathbb{N}.$$ □

Lemma 3.5.3 is a direct generalization of [Charina, 2012, Lemma 4.5]. It connects the computation of the norms of transition matrices, defined in 3.2.6, with the computation of the norms of subdivision operators applied to the δ-sequence.

Lemma 3.5.3. *Let $k \in \mathbb{N}_0$ and $\mathcal{S}$ be a finite set of subdivision operators whose masks satisfy sum rules of order $k+1$. If there exists a finite set $\Omega \subseteq \mathbb{Z}^s$ such that*

(i) $\ell(\Omega)$ is invariant under the transition operators in $\mathcal{T}$,

(ii) $V_k(\Omega) = \tilde{V}_k(\Omega)$ and

(iii) $X_\mu \neq \emptyset$ for all $|\mu| = k+1$, where

$$X_\mu = \left\{ \beta \in \mathbb{Z}^s : \operatorname{supp} \tilde{\nabla}^\mu \, \delta(\cdot - \beta) \subseteq \Omega \right\}, \tag{3.5.1}$$

then for all $j \in \{1, \dots, J\}$

$$\binom{k+1}{\lfloor \frac{k+1}{2} \rfloor}^{-1} \left|\left| \nabla^{k+1} S_j \delta \right|\right|_\infty \leq \max_{d \in D_j} \left|\left| T_{d,j,\Omega}|_{V_{k-1}(\Omega)} \right|\right|_\infty \leq \left|\left| \mathcal{V}^+ \right|\right|_\infty \binom{s+k}{s-1} \#\Omega \left|\left| \nabla^{k+1} S_j \delta \right|\right|_\infty,$$

where $\mathcal{V}^+$ is a matrix defined by (3.5.3) *and depending only on Ω and k.*

Proof. First, we show the right inequality. Let $k \in \mathbb{N}_0$ and $v \in V_k(\Omega)$. We view v as a vector in $\mathbb{R}^{\#\Omega}$. By Definition 3.4.14 of $\tilde{V}_k(\Omega)$ and assumption *(ii)*, there exist $v_{\mu,\beta} \in \mathbb{R}$, with $\mu \in \mathbb{N}_0^s$, $|\mu| = k+1$, $\beta \in X_\mu$, such that

$$v = \sum_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \sum_{\beta \in X_\mu} v_{\mu,\beta} \tilde{\nabla}^\mu \, \delta(\cdot - \beta). \tag{3.5.2}$$

If we define $\mathcal{V}$ as the matrix whose columns are the finite sequences $\tilde{\nabla}^\mu \, \delta(\cdot - \beta)$, $\beta \in X_\mu$, $\mu \in \mathbb{N}_0^s$, $|\mu| = k+1$, then we can compute a vector $\tilde{v}$, whose entries are numbers $v_{\mu,\beta}$, $\mu \in \mathbb{N}_0^s$, $|\mu| = k+1$, $\beta \in X_\mu$, such that (3.5.2) holds, by

$$\tilde{v} = \mathcal{V}^+ v, \tag{3.5.3}$$

where $\mathcal{V}^+$ is a pseudo-inverse of $\mathcal{V}$. It follows that we can bound the modulus of the numbers $v_{\mu,\beta}$ by

$$\max_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \max_{\beta \in X_\mu} |v_{\mu,\beta}| = ||\tilde{v}||_\infty = \left|\left| \mathcal{V}^+ v \right|\right|_\infty \leq \left|\left| \mathcal{V}^+ \right|\right|_\infty \cdot ||v||_\infty, \quad v \in V_k(\Omega). \tag{3.5.4}$$

With $T_d = T_{d,j,\Omega}$ for some $d \in D_j$, $j \in \{1, \dots, J\}$, using that $\#X_\mu \leq \#\Omega$ and (3.5.4),

$$\begin{aligned}
\left|\left| T_d|_{V_k(\Omega)} \right|\right|_\infty &= \max_{\substack{v \in V_k(\Omega) \\ ||v||_\infty = 1}} \left|\left| T_d \sum_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \sum_{\beta \in X_\mu} v_{\mu,\beta} (\tilde{\nabla}^\mu \, \delta)(\cdot - \beta) \right|\right|_\infty \\
&\leq \max_{\substack{v \in V_k(\Omega) \\ ||v||_\infty = 1}} ||\tilde{v}||_\infty \sum_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \sum_{\beta \in X_\mu} \left|\left| T_d (\tilde{\nabla}^\mu \, \delta)(\cdot - \beta) \right|\right|_\infty \\
&\leq \left|\left| \mathcal{V}^+ \right|\right|_\infty \binom{s+k}{s-1} \#\Omega \max_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \max_{\beta \in X_\mu} \left|\left| T_d (\tilde{\nabla}^\mu \, \delta)(\cdot - \beta) \right|\right|_\infty.
\end{aligned} \tag{3.5.5}$$

By the definition of T_d, we get for $\beta \in X_\mu$, $|\mu| = k+1$,

$$\left|\left|(T_d\tilde{\nabla}^\mu\,\delta)(\,\cdot\,-\beta)\right|\right|_\infty = \max_{\alpha\in\Omega}\left|\sum_{\gamma\in\Omega} a_j(M_j\alpha-\gamma+d)\tilde{\nabla}^\mu\,\delta(\gamma-\beta)\right|_\infty . \tag{3.5.6}$$

The value of (3.5.6) does not change if we replace $\alpha \in \Omega$ by $\alpha \in \mathbb{Z}^s$, due to assumption (i). The value of (3.5.6) also does not change if we take $\gamma \in \mathbb{Z}^s$, since $\operatorname{supp}\tilde{\nabla}^\mu\,\delta(\,\cdot\,-\beta) \subseteq \Omega$. Thus, for $\beta \in X_\mu$,

$$\begin{aligned}\left|\left|T_d\tilde{\nabla}^\mu\,\delta(\,\cdot\,-\beta)\right|\right|_\infty &= \max_{\alpha\in\mathbb{Z}^s}\left|\sum_{\gamma\in\mathbb{Z}^s} a_j(M_j\alpha-\gamma+d)\tilde{\nabla}^\mu\,\delta(\gamma-\beta)\right| \\ &= \left|\left|(\tilde{\nabla}^\mu\,a_j)(M_j\,\cdot\,+d-\beta)\right|\right|_\infty = \left|\left|(\tilde{\nabla}^\mu\,S\delta)(M\,\cdot\,+d-\beta)\right|\right|_\infty .\end{aligned} \tag{3.5.7}$$

Now, with $C_1 = ||\mathcal{V}^+||_\infty \binom{s+k}{s-1}\#\Omega$, using the fact that D_j is a digit set of M_j and combining (3.5.5) and (3.5.7) we obtain the first estimate

$$\begin{aligned}\max_{d\in D_j}\left|\left|T_d|_{V_k(\Omega)}\right|\right|_\infty &\le C_1 \max_{d\in D_j}\max_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k+1}}\max_{\beta\in X_\mu}\left|\left|T_d\tilde{\nabla}^\mu\,\delta(\,\cdot\,-\beta)\right|\right|_\infty \\ &= C_1 \max_{d\in D_j}\max_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k+1}}\max_{\beta\in X_\mu}\left|\left|(\tilde{\nabla}^\mu\,S_j\delta)(M_j\,\cdot\,+d-\beta)\right|\right|_\infty \\ &= C_1 \max_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k+1}}\left|\left|\tilde{\nabla}^\mu\,S_j\delta\right|\right|_\infty \\ &= C_1\left|\left|\nabla^{k+1}\,S_j\delta\right|\right|_\infty .\end{aligned}$$

To show the reverse inequality, take $\mu \in \mathbb{N}_0^s$, $|\mu| = k+1$ and $\beta \in X_\mu$.Note that β exists, due to assumption (iii). Thus, by Lemma 3.5.2,

$$\begin{aligned}||T_d|_{V_k}||_\infty &= \max_{\substack{v\in V_k(\Omega)\\v\neq 0}}\frac{||T_dv||_\infty}{||v||_\infty} \\ &\ge \frac{\left|\left|T_d\tilde{\nabla}^\mu\,\delta(\,\cdot\,-\beta)\right|\right|_\infty}{\left|\left|\tilde{\nabla}^\mu\,\delta(\,\cdot\,-\beta)\right|\right|_\infty} \\ &\ge \binom{k+1}{\lfloor\frac{k+1}{2}\rfloor}^{-1}\left|\left|T_d\tilde{\nabla}^\mu\,\delta(\,\cdot\,-\beta)\right|\right|_\infty .\end{aligned}$$

Taking the maximum over $d \in D_j$ and using (3.5.7) we obtain

$$\begin{aligned}\max_{d\in D_j}||T_d|_{V_k}||_\infty &\ge \binom{k+1}{\lfloor\frac{k+1}{2}\rfloor}^{-1}\max_{d\in D_j}\max_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k+1}}\left|\left|(\tilde{\nabla}^\mu\,S_j\delta)(M_j\,\cdot\,+d-\beta)\right|\right|_\infty \\ &= \binom{k+1}{\lfloor\frac{k+1}{2}\rfloor}^{-1}\left|\left|\nabla^{k+1}\,S_j\delta\right|\right|_\infty .\end{aligned}$$

□

Lemma 3.5.4 is a direct generalization of [Charina, 2012, Proposition 4.6]. It connects Lemma 3.5.1 with Lemma 3.5.3.

Lemma 3.5.4. *Let $k \in \mathbb{N}_0$. If $\mathcal{S}$ is a finite set of subdivision operators whose masks satisfy sum rules of order $k+1$ and $\mathcal{S}'$ is a set of corresponding difference subdivision operators of order $k+1$, then for all $j \in \{1, \dots, J\}$, there exists $K \subseteq \mathbb{Z}^s$ finite and depending only on $\mathcal{S}$, such that*

$$\left|\left|\mathcal{W}^+\right|\right|_\infty \#K \left|\left|S'_j|_{\nabla^{k+1}}\right|\right|_\infty \leq \left|\left|\nabla^{k+1} S_j\delta\right|\right|_\infty \leq \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor} \left|\left|S'_j|_{\nabla^{k+1}}\right|\right|_\infty,$$

where $\mathcal{W}^+$ is a matrix defined by (3.5.8) *and depending only on K and k.*

Proof. We show the right inequality first. By Lemma 3.5.2 (ii), we have $\left|\left|\nabla^{k+1}\delta\right|\right|_\infty = \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor}$, and thus

$$\sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s)\\ ||\nabla^{k+1}c||_\infty=1}} \left|\left|S'_j\nabla^{k+1}c\right|\right|_\infty = \sup_{\substack{c\in\ell_\infty(\mathbb{Z}^s)\\ ||\nabla^{k+1}c||=1}} \left|\left|\nabla^{k+1}S_jc\right|\right|_\infty \geq \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor}^{-1} \left|\left|\nabla^{k+1}S_j\delta\right|\right|_\infty.$$

For the other inequality, by Lemma 3.5.1, there exists a finite set $K \subseteq \mathbb{Z}^s$, only depending on $\mathcal{S}$, and a maximizing sequence $\tilde{c} \in \ell_\infty(K)$, $\left|\left|\nabla^{k+1}\tilde{c}\right|\right|_\infty = 1$ such that

$$\left|\left|S'_j|_{\nabla^{k+1}}\right|\right|_\infty = \max_{\alpha\in M_j[0,1)^s\cap\mathbb{Z}^s} \left|\left|S'_j\nabla^{k+1}\tilde{c}(\alpha)\right|\right|_\infty.$$

We view $\tilde{c}$ as a vector in $\mathbb{R}^K$ and $\nabla^{k+1}\tilde{c}$ as vector in $\mathbb{R}^{\binom{s+k}{s-1}} \times \mathbb{R}^{K+\{0,\dots,k+1\}^s}$. Let $\mathcal{W}$ be the restriction of the $(k+1)^{st}$ backwards difference operator ∇^{k+1} to finite vectors, precisely,

$$\mathcal{W}(\,\cdot\,|_{\mathbb{R}^K}) = (\nabla^{k+1}\,\cdot\,)|_{\mathbb{R}^{\binom{s+k}{s-1}}\times\mathbb{R}^{K+\{0,\dots,k+1\}^s}}. \tag{3.5.8}$$

Thus, $||\mathcal{W}\tilde{c}||_\infty = \left|\left|\nabla^{k+1}\tilde{c}\right|\right|_\infty = 1$ and we can bound $||\tilde{c}||_\infty$ by

$$\left|\left|\tilde{c}\right|\right|_\infty = \left|\left|\mathcal{W}^+\mathcal{W}\tilde{c}\right|\right|_\infty \leq \left|\left|\mathcal{W}^+\right|\right|_\infty \left|\left|\mathcal{W}\tilde{c}\right|\right|_\infty \leq \left|\left|\mathcal{W}^+\right|\right|_\infty. \tag{3.5.9}$$

Therefore,

$$\begin{aligned}
\left|\left|S'_j|_{\nabla^{k+1}}\right|\right|_\infty &= \max_{\alpha\in\left(M_j[0,1)^s\right)\cap\mathbb{Z}^s} \left|\left|S'_j \sum_{\beta\in K} \nabla^{k+1}\delta(\alpha-\beta)\tilde{c}(\beta)\right|\right|_\infty \\
&\leq \left|\left|\mathcal{W}^+\right|\right|_\infty \max_{\alpha\in\left(M_j[0,1)^s\right)\cap\mathbb{Z}^s} \sum_{\beta\in K} \left|\left|S'_j\nabla^{k+1}\delta(\alpha-\beta)\right|\right|_\infty \\
&\leq \left|\left|\mathcal{W}^+\right|\right|_\infty \#K \max_{\alpha\in\left(M_j[0,1)^s-K\right)\cap\mathbb{Z}^s} \left|\left|S'_j\nabla^{k+1}\delta(\alpha)\right|\right|_\infty \\
&\leq \left|\left|\mathcal{W}^+\right|\right|_\infty \#K \left|\left|\nabla^{k+1}S_j\delta\right|\right|_\infty.
\end{aligned}$$

□

Now, the proof of the main theorem 3.4.17 just consists of combining these three lemmata.

Proof of Theorem 3.4.17. By Definition 3.4.12 of the RSR and Definition 2.1.1 of the JSR, we have to show that there exist constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$

$$C_1 \sup_{S'_j\in\mathcal{S}'} \left|\left|S'_{j_n}\cdots S'_{j_1}|_{\nabla^{k+1}}\right|\right| \leq \max_{T_j\in\mathcal{T}} \left|\left|T_{j_n}\cdots T_{j_1}|_{V_k(\Omega)}\right|\right|_\infty \leq C_2 \sup_{S'_j\in\mathcal{S}'} \left|\left|S'_{j_n}\cdots S'_{j_1}|_{\nabla^{k+1}}\right|\right|.$$

The constants, by Lemma 3.5.3 and 3.5.4, are

$$C_1 = \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor}^{-1} ||\mathcal{W}^+||_\infty \#K \quad \text{and}$$
$$C_2 = ||\mathcal{V}^+||_\infty \binom{s+k}{s-1} \#\Omega \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor},$$

where $K \subseteq \mathbb{Z}^s$ is a finite set depending only on $\mathcal{S}$ and $\mathcal{V}^+$ and $\mathcal{W}^+$ are matrices only depending on $\mathcal{S}$ and k and defined in (3.5.3) and (3.5.8), respectively, □

Example 3.5.5 shows that already in the univariate, stationary case the assumptions $V_k(\Omega) = \tilde{V}_k(\Omega)$ and $X_\mu \neq \emptyset$ for all $|\mu| = k+1$, in Theorem 3.4.17 are crucial.

Example 3.5.5. We consider the stationary subdivision scheme with the dilation matrix $M = 2$ and the mask

$$a = \frac{1}{2}\begin{bmatrix}\mathbf{1} & 0 & 0 & 2 & 0 & 0 & 1\end{bmatrix}^T.$$

This subdivision scheme does not converge, as we will prove later, although there is a continuous, piecewise linear, compactly supported on $[0, 6]$, solution ϕ of the corresponding refinement equation

$$\phi(x) = \frac{1}{2}\phi(2x) + \phi(2x-3) + \frac{1}{2}\phi(2x-6). \tag{3.5.10}$$

Indeed, the function

$$\phi(x) = \begin{cases} x/3 & \text{if } x \in [0,3], \\ 2 - x/3 & \text{if } x \in [3,6] \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

solves (3.5.10). We show next that the subdivision scheme is not convergent.

For illustration purposes, we choose the digit set $D = \{0, 3\}$. Algorithm 3.2.9 generates the set $\Omega_C = \{0, 3\}$ and, by Definition 3.4.14, $\dim V_0(\Omega_C) = 2 > \dim \tilde{V}_0(\Omega_C) = 0$. Furthermore, $X_{(1)} = \emptyset$. Thus, Ω_C fails to fulfil the assumptions 3.4.17 (ii) and (iii). The set $\Omega_0(V)$, with the properties $\dim V_0(\Omega_V) = \dim \tilde{V}_0(\Omega_V)$ and $X_{(1)} \neq \emptyset$. can be chosen, in this case, to be $\Omega_V = \{-2, \ldots, 5\}$. We make this choice for simplicity reasons, the set Ω_V from Proposition 3.4.21 would be of size 18. For $\Omega' = \Omega_V \backslash \Omega_C = \{-2, -1, 1, 2, 4, 5\}$, the corresponding transition matrices T_{d,Ω_V}, $d \in D$, have the following block form

$$T_{0,\Omega_V} = \frac{1}{2}\begin{bmatrix} T_{0,\Omega_C} & 0 \\ 0 & T_{0,\Omega'} \end{bmatrix} \quad \text{and} \quad T_{3,\Omega_V} = \frac{1}{2}\begin{bmatrix} T_{3,\Omega_C} & 0 \\ 0 & T_{3,\Omega'} \end{bmatrix},$$

with

$$T_{0,\Omega_C} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad T_{3,\Omega_C} = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix},$$

and

$$T_{0,\Omega'} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad T_{3,\Omega'} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Lemma 3.2.8 (ii), the space $\ell(\Omega_V)$ is $\mathcal{T}$ invariant. Thus, by Theorems 3.4.13 and 3.4.17, due to $\dim V_0(\Omega_V) = \dim \tilde{V}_0(\Omega_V)$, $X_{(1)} \neq \emptyset$ and $\mathrm{JSR}(\{T_{d,\Omega_V}|_{V_0(\Omega_V)} : d \in D\}) = 1$, we get the correct

answer that the scheme is not convergent. On the contrary, $\mathrm{JSR}(\{T_{d,\Omega_C}|_{V_0(\Omega_C)} : d \in D\}) = \frac{1}{2}$ is misleading. Here, we used the modified invariant polytope algorithm from Chapter 4 for our computations. △

Example 3.5.6. We present a full working example for Matlab how to check the convergence properties of a multiple subdivision scheme. We consider again the bivariate multiple scheme defined in Example 3.2.4.

We first define the masks `a1=1/3*[1 2 3 2 1]`, `a2=a1`, the dilation matrices `M1=[1 1;1 -2]`, `M2=[2 -1;1 -2]` the subdivision operators `S1=getS({a, M1})`, `S2=getS({a, M2})`, and the set `S=[S1; S2]`. The function `getS` computes the digit sets $D_1 = \left\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right\}$ and $D_2 = \left\{\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right\}$ which we can view with `D1=S1{3}` and `D2=S2{3}`. The command `tjsr({M1^(-1),M2^(-1)})` checks that the dilation matrices are indeed jointly expanding, since the returned value is less than one. The function `tjsr` [Mejstrik, 2018b] computes the joint spectral radius of a finite set of matrices. It is our implementation of the modified invariant polytope algorithm; described in detail in Chapter 4. Next we construct the set Ω_C and the transition matrices by `[T,OmC]=transitionmatrix(S)` and the spaces $V_0(\Omega_C)$ and $\tilde{V}_0(\Omega_C)$. To check that these spaces are equal we call `dimVVt(OmC)` and infer that $\dim V_0(\Omega_C) = \dim \tilde{V}_0(\Omega_C) = 39$. We check by hand that $X_{(1,0)}$ and $X_{(0,1)}$ are non-empty. Next we construct the space $V_0(\Omega_C)$ and restrict the transition matrices to this space by `V0=constructV(OmC,0)` and `TV0=restrictmatrix(T,V0)`. Finally we compute the joint spectral radius of the restricted transition matrices $T_{d,j,\Omega}|_{V_0(\Omega_C)}$, $j = 1, 2$, $d \in D_j$, by `tjsr(TV0,'maxsmpdepth',2)`. The option `'maxsmpdepth',2` is useful in this case and its effect is described in the manual [Mejstrik, 2018b]. Without this option the function runs much slower, but still returns

$$\begin{aligned}\mathrm{JSR}\left(\left\{T_{d,j,\Omega_C}|_{V_0(\Omega_C)} : j = 1, 2,\ d \in D_j\right\}\right) &= \rho\left(T_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],1,\Omega_C} T_{\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right],2,\Omega_C}|_{V_0(\Omega_C)}\right)^{1/2} \\ &= \sqrt{\frac{1+\sqrt{2}}{3}} \simeq 0.89707.\end{aligned}$$

Therefore, $\mathtt{S}^{\mathbb{N}}$ is convergent. For convenience of the reader, the raw source code is given in Figure 3.10. △

```
a1 = 1/3*[ 1 2 3 2 1 ]; a2 = a1;
M1 = [ 1  1; 1 -2 ];
M2 = [ 2 -1; 1 -2 ];
S1 = getS( { a1, M1 } )
S2 = getS( { a2, M2 } )
S = [ S1; S2 ]
tjsr( {S1{2}^-1, S2{2}^-1} )
[ T, OmC ] = transitionmatrix(S)
dimVVt( OmC )
V0 = constructV( OmC, 0 )
TV0 = restrictmatrix( T, V0 )
tjsr( TV0, 'maxsmpdepth',2 )
```

Figure 3.10: Listing for Example 3.5.6.

3.6 Appendix for Chapter 3

Algorithm 3.6.1. Simple implementation of the invariant Omega algorithm 3.2.9 in Matlab.

```
function [ Om ] = constructOmega( S, Om )
% S: cell array of subdivision schemes. Each row consists of a, M and D.
% Om: (Optional) the starting set
% Ex: a=1/3*[1 2 3 2 1]; M1=[2 -1;1 -2]; M2=[1 1;1 -2]; D=[0 1 2;0 0 0];
%     constructOmega({a, M1, D; a, M2, D})
    a=S(:,1); M=S(:,2); D=S(:,3);         %extract the sets a, M and D
    J=numel(a);                           %number of subdivision operators
    dim=size(M{1},1);                     %the dimension
    if(nargin==1); Om=zeros(dim,1); end   %if Omega is not given, set it to zero
    while(true)
        sizebefore=size(Om,2);            %used to check if elements were added
        for j=1:J                         %iterate through all subdiv. operators
            OmN=M{j}\setplus(supp(a{j},dim),Om,-D{j});
                                          %compute new possible entries
            OmN=round(OmN(:,sum(abs(OmN-round(OmN)),1)<.5/abs(det(M{j}))));
                                          %round to integers
            Om=unique([Om OmN]','rows')'; %remove duplicates
        end
        if(size(Om,2)==sizebefore); break; end
                                          %if no elements were added, terminate
    end

function [ X ] = setplus( varargin )
% setplus(A,B) = { x=a+b : a in A, b in B},
% Note: The function operates column wise
% Ex: setplus([1 2; 1 0],[0 -1;-1 -1]); %Output: [0 1 1 2;0 -1 0 -1]
    sze=size(varargin,2);                 %number of sets
    X=varargin{sze};                      %the output set
    for j=sze-1:-1:1                      %iterate through all sets
        A=varargin{j};                    %the set to be added
        X=repmat(A,1,size(X,2))+reshape(repmat(X,size(A,2),1),size(A,1),[]);
                                          %add the set
        X=unique(X','rows')';             %remove duplicates
    end

function [ L ] = supp( a, dim )
% Returns the support of an array.
% The first entry is assumed to have index (0,0,...,0)
% Ex: supp([1 1;0 1],2) %Output: [0 0 1;0 1 1];
    L=zeros(dim,nnz(a));                  %output variable
    CO=cell(1,dim);                       %dummy-variable
    k=1;                                  %index-variable for D
    for j=1:numel(a)                      %iterate through all elements of the masks
        if(a(j)~=0)                       %if the element is nonzero, save the indices
            [CO{:}]=ind2sub(size(a),j);   %get the indices
            L(:,k)=[CO{:}]'-1;            %add converted cell to vector
            k=k+1;                        %increase counter
        end
    end
```

Proof of Example 3.1.5. For the subdivision operator $S_1 = (a_1, M_1)$, with the mask $a_1 = \frac{1}{4}\begin{bmatrix}-\mathbf{1} & -2 & 2 & 6 & 3\end{bmatrix}^T$, the dilation matrix $M_1 = 2$ and the digit set $D = \{0, 1\}$, the space $\ell(\Omega)$ with $\Omega = \{0, 1, 2, 3\}$ is $\mathcal{T}$ invariant. By [Daubechies and Lagarias, 1992b, Theorem 2.2], the Hölder regularity α of the basic limit function of the corresponding subdivision scheme is given by the JSR of the transition matrices restricted to the space $V_0(\Omega)$ and computes to

$$\alpha = -\log_2 \operatorname{JSR}\left(\left\{\frac{1}{4}\begin{bmatrix}-4 & -3 & -2\\ 3 & -1 & -5\\ 0 & 3 & 6\end{bmatrix}, \frac{1}{4}\begin{bmatrix}-4 & -8 & -7\\ 3 & 6 & 2\\ 0 & 0 & 3\end{bmatrix}\right\}\right) = -\log_2 \frac{3}{4} \simeq 0.4150.$$

For the subdivision operator $S_2 = (a_2, M_2)$, with the mask $a_1 = \frac{1}{4}\begin{bmatrix}\mathbf{3} & 2 & 2 & 2 & -1\end{bmatrix}^T$, the dilation matrix $M_1 = 2$ and digit set $D = \{0, 1\}$, the space $\ell(\Omega)$, with $\Omega = \{0, 1, 2, 3\}$, is $\mathcal{T}$ invariant. As above, the Hölder regularity α of the basic limit function is

$$\alpha = -\log_2 \operatorname{JSR}\left(\left\{\frac{1}{4}\begin{bmatrix}0 & 0 & 1\\ -1 & 2 & 2\\ 0 & 0 & -1\end{bmatrix}, \frac{1}{4}\begin{bmatrix}0 & 1 & -2\\ 3 & 3 & 3\\ 0 & -1 & 2\end{bmatrix}\right\}\right) = -\log_2 \frac{3}{4} \simeq 0.4150.$$

For the multiple subdivision scheme $\mathcal{S} = \{S_1, S_2\}$, the joint spectral radius of the restricted transition matrices computes to $\frac{5}{4}$. Therefore, $\mathcal{S}^{\mathbb{N}}$ is not convergent. □

4 Joint spectral radius

Ah ha.

(V. Yu. Protasov)

Albeit the characterization of the convergence of subdivision schemes in terms of the joint spectral radius is elegant, the numerical computation of the joint spectral radius still causes big problems. In several papers, Guglielmi and Protasov [2013, 2016] made a breakthrough in the problem of the joint spectral radius computation, developing the invariant polytope algorithm which for a large class of matrices finds the exact value of the joint spectral radius. This algorithm found many applications in (seemingly unrelated) mathematical applications, for example for computing the regularity of wavelets and subdivision schemes [Daubechies and Lagarias, 1992b], the capacity of codes [Moision, Orlitsky and Siegel, 2001], the stability of linear switched systems [Gurvits, 1995] or in connection with the Euler partition function [Protasov, 2000].

In Section 4.3 we introduce the modified Gripenberg algorithm. This is a new, fast algorithm for computing good lower bounds for the joint spectral radius. The modified Gripenberg algorithm finds in most cases the exact value of the joint spectral radius in less than 5 seconds.

In Section 4.4 we propose a modification of the invariant polytope algorithm, parallelising it, making it roughly three times faster and suitable for higher dimensions. Furthermore, in cases where the modified invariant polytope algorithm fails to find the exact value of the JSR, the modified invariant polytope algorithm can return upper bounds for the JSR which are proven to be correct. The original invariant polytope algorithm may return wrong bounds. The modified version works for most matrix families of dimensions up to 25 (compared to 20 for the original algorithm); for non-negative matrices the dimension is up to three thousand (compared to one thousand for the original algorithm).

Corresponding examples and statistics of numerical results are provided in Section 4.5.

4.1 Definitions and properties

We begin this chapter by recalling the definition of the *joint spectral radius* (JSR). Given a finite set of matrices $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$, $J \in \mathbb{N}$, its JSR is defined by

$$\mathrm{JSR}(\mathcal{A}) = \lim_{n\to\infty} \max_{A_j\in\mathcal{A}} ||A_{j_n} \cdots A_{j_1}||^{1/n}. \tag{4.1.1}$$

The JSR of a finite set of matrices is a quantity which describes the maximal asymptotic growth rate of the norms of products of matrices from this set (with repetitions permitted). If the set $\mathcal{A}$ consists of only one matrix, the JSR reduces to the classical *spectral radius* ρ of a matrix. Given $A \in \mathbb{R}^{s\times s}$, by Gelfand's formula,

$$\rho(A) = \lim_{n\to\infty} ||A^n||^{1/n}, \tag{4.1.2}$$

which makes the connection between the spectral radius and the joint spectral radius immediately clear. Before we go on, we prove that the JSR exists.

Lemma 4.1.1. *If $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$ is a finite set of matrices, then*

(*i*) $\mathrm{JSR}(\mathcal{A})$ *exists and is equal to* $\inf_{n\in\mathbb{N}} \max_{A_j\in\mathcal{A}} ||A_{j_n}\cdots A_{j_1}||^{1/n}$ *and*

(*ii*) $\mathrm{JSR}(\mathcal{A}) = \inf_{||\cdot||} \max_{A_j\in\mathcal{A}} ||A_j||$.

The proof for this Lemma is taken from [Jungers, 2009, Proposition 1.2 and Proposition 1.4].

Proof. First note that, due to the equivalence of norms in $\mathbb{R}^s$, the definition of the JSR in (4.1.1) does not depend on the norm.

(*i*) Let $||\cdot||$ be a sub-multiplicative norm and for $n \in \mathbb{N}$ let $r_n = \max_{A_j\in\mathcal{A}} ||A_{j_n}\cdots A_{j_1}||$. It follows that $(r_n)_{n\in\mathbb{N}}$ is a sub-multiplicative sequence, i.e. $r_{m+n} \le r_m r_n$, and $r_n > 0$ for all $m, n \in \mathbb{N}$ (When $r_n = 0$ for any $n \in \mathbb{N}$, the proof is trivial). Therefore, by Fekete's Lemma $\lim_{n\to\infty} r_n^{1/n} = \inf_{n\in\mathbb{N}} r_n^{1/n}$, which proves the claim.

(*ii*) Let $\epsilon > 0$ and define $\tilde{\mathcal{A}} = \left\{(\mathrm{JSR}(\mathcal{A}) + \epsilon)^{-1} A_j : A_j \in \mathcal{A}\right\}$. By (4.1.1), the norms of all products of the form $\tilde{A}_{j_n}\cdots\tilde{A}_{j_1}$, $\tilde{A}_j \in \tilde{\mathcal{A}}$, are uniformly bounded. Thus, we can define a vector norm $||\cdot||_\epsilon : \mathbb{R}^s \to \mathbb{R}$, $||x||_\epsilon = \max\left\{\|x\|_2, \|\tilde{A}_{j_n}\cdots\tilde{A}_{j_1}x\|_2 : \tilde{A}_j \in \tilde{\mathcal{A}},\ n \in \mathbb{N}\right\}$. The matrix norm induced by this vector norm fulfils $\|\tilde{A}_j\|_\epsilon \le 1$ for all $\tilde{A}_j \in \tilde{\mathcal{A}}$. Indeed, for $\tilde{A}_j \in \tilde{\mathcal{A}}$,

$$\begin{aligned}\left|\left|\tilde{A}_j\right|\right|_\epsilon &= \max_{||x||_\epsilon=1} \left|\left|\tilde{A}_j x\right|\right|_\epsilon = \max_{||x||_\epsilon=1} \max\left\{\left|\left|\tilde{A}x\right|\right|_2, \left|\left|\tilde{A}_{j_n}\cdots\tilde{A}_{j_1}\tilde{A}x\right|\right|_2 : \tilde{A}_j \in \tilde{\mathcal{A}},\ n \in \mathbb{N}\right\}\\ &\le \max_{||x||_\epsilon=1} ||x||_\epsilon = 1.\end{aligned}$$

Therefore, $||A_j||_\epsilon \le \mathrm{JSR}(\mathcal{A}) + \epsilon$ for all $A_j \in \mathcal{A}$. □

For completeness, we give a self contained proof of Fekete's Lemma in Section 4.6.

The computation of the JSR is a notoriously hard problem. Even for non-negative matrices with rational coefficients this problem is NP-hard [Blondel and Tsitsiklis, 1997]. Moreover, the question whether or not $\mathrm{JSR}(\mathcal{A}) \le 1$ for a given set $\mathcal{A}$ is algorithmically undecidable [Blondel and Tsitsiklis, 2000]. Berger and Wang [1992] proved that the JSR can be computed using the spectral radii of the matrix products. If $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$, then

$$\mathrm{JSR}(\mathcal{A}) = \limsup_{\substack{n\to\infty\\ A_j\in\mathcal{A}}} \rho(A_{j_n}\cdots A_{j_1})^{1/n}. \tag{4.1.3}$$

Most algorithms which try to compute or to approximate the JSR make use of the following inequality.

Theorem 4.1.2 ([Daubechies and Lagarias, 1992a, Lemma 3.1])**.** *If $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$ and $||\cdot||$ is a sub-multiplicative norm, then*

$$\max_{A_j\in\mathcal{A}} \rho\left(A_{j_k}\cdots A_{j_1}\right)^{1/k} \le \mathrm{JSR}(\mathcal{A}) \le \max_{A_j\in\mathcal{A}} ||A_{j_k}\cdots A_{j_1}||^{1/k}, \quad k \in \mathbb{N}. \tag{4.1.4}$$

Proof. Let $||\cdot||$ be any sub-multiplicative norm and $k \in \mathbb{N}$. We first prove the left inequality in (4.1.4). By (4.1.2), we get for all $m \in \mathbb{N}$,

$$\begin{aligned}\rho(A_{j_k}\cdots A_{j_1})^{1/k} &= \rho((A_{j_k}\cdots A_{j_1})^m)^{1/km} \le ||(A_{j_k}\cdots A_{j_1})^m||^{1/km} \le \max_{A_j\in\mathcal{A}} ||A_{j_{km}}\cdots A_{j_1}||^{1/km}\\ &\le \lim_{m\to\infty} \max_{A_{km}\in\mathcal{A}} ||A_{j_{km}}\cdots A_{j_1}||^{1/km} = \mathrm{JSR}(\mathcal{A}).\end{aligned}$$

The right inequality follows directly from Fekete's Lemma 4.6.3. Indeed, for $k \in \mathbb{N}$,

$$\mathrm{JSR}(\mathcal{A}) = \lim_{n\to\infty} \max_{A_j\in\mathcal{A}} ||A_{j_n}\cdots A_{j_1}||^{1/n} = \inf_{n\in\mathbb{N}} \max_{A_j\in\mathcal{A}} ||A_{j_n}\cdots A_{j_1}|| \ge \max_{A_j\in\mathcal{A}} ||A_{j_k}\cdots A_{j_1}||^{1/k}. \quad \square$$

In general, one cannot deduce upper bounds of the JSR from the spectral radii of the individual matrices.

Example 4.1.3. For $N \in \mathbb{N}$, the two matrices $A_1 = \begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}$ both have spectral radius zero, $\rho(A_1) = \rho(A_2) = 0$, while Theorem 4.1.2 implies that $\mathrm{JSR}(\{A_1, A_2\}) \geq \rho(A_1 A_2)^{1/2} = N$. △

Using Theorem 4.1.2 it is easy to show that the JSR of the matrices in Example 4.1.3 equals N. Thus, the JSR of these matrices is attained by the spectral radius of a finite product of these matrices. This brings us to the concept of *spectral maximizing products*.

Definition 4.1.4. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \ldots, J\}$. For the product $\Pi_1 = A_{j_k} \cdots A_{j_1}$, $A_j \in \mathcal{A}$, $k \in \mathbb{N}$,

(*i*) we denote the *length* k of the product by $l(\Pi_1)$, or shortly by $l(1)$ and

(*ii*) we call the number $\rho(\Pi_1)^{1/l(1)} = \rho(A_{j_k} \cdots A_{j_1})^{1/k}$ the products *averaged spectral radius* and

(*iii*) we call the number $||\Pi_1||^{1/l(1)} = ||A_{j_k} \cdots A_{j_1}||^{1/k}$ the products *averaged norm*.

Definition 4.1.5. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \ldots, J\}$. A *spectral maximizing product (s.m.p.)* is any matrix product of matrices in $\mathcal{A}$ whose averaged spectral radius equals the joint spectral radius of $\mathcal{A}$.

Example 4.1.6. Let $\mathcal{A} = \{A_1, A_2\}$ with $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Computing the averaged spectral radii and norms of products of matrices of length one and two from the set $\mathcal{A}$, we get

$$\rho(A_1)^{1/1} = 1, \qquad ||A_1||_2^{1/1} = \frac{\sqrt{5}+1}{2},$$
$$\rho(A_1A_2)^{1/2} = \frac{\sqrt{5}+1}{2}, \qquad ||A_1A_2||_2^{1/2} = \frac{\sqrt{5}+1}{2}, \qquad \left|\left|A_1^2\right|\right|_2^{1/2} = \sqrt{1+\sqrt{2}}.$$

Note that, due to the symmetry of the matrices A_1 and A_2, we do not need to compute all possible products. By Theorem 4.1.2, $\mathrm{JSR}(\mathcal{A}) = \frac{\sqrt{5}+1}{2}$ and A_1A_2 is an s.m.p..

In general, the averaged norms do not attain the value of the JSR. For example, for the ∞-norm we get for all products of length less or equal than four,

$$||A_1||_\infty^{1/1} = 2,$$
$$||A_1A_1||_\infty^{1/2} = 3^{1/2}, \qquad ||A_1A_2||_\infty^{1/2} = 3^{1/2},$$
$$||A_1A_1A_1||_\infty^{1/3} = 4^{1/3}, \qquad ||A_1A_1A_2||_\infty^{1/3} = 5^{1/3},$$
$$||A_1A_1A_1A_1||_\infty^{1/4} = 5^{1/4}, \qquad ||A_1A_1A_1A_2||_\infty^{1/4} = 7^{1/4}, \qquad ||A_1A_2A_1A_2||_\infty^{1/4} = 8^{1/4},$$
$$||A_1A_1A_2A_2||_\infty^{1/4} = 7^{1/4}.$$

The maximum over all averaged ∞-norms of length $k \in \{1, \ldots, 50\}$ is plotted in Figure 4.1. △

Bousch and Mairesse [2002] proofed that not all sets of matrices posses an s.m.p.. Hare, Morris, Sidorov and Theys [2011] gave the first explicit counterexample of a set which does not posses an s.m.p.. This means that there exist sets of matrices such that the averaged spectral radius of every finite product is strictly less than the JSR.

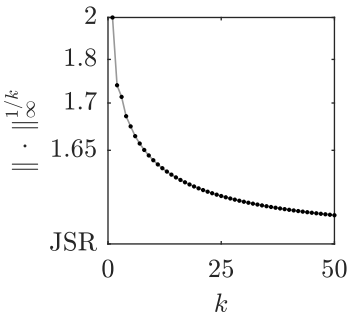


Figure 4.1: Maximum of averaged norms of products of length $k \in \mathbb{N}$ for the matrices defined in Example 4.1.6. JSR $= \frac{\sqrt{5}+1}{2} \simeq 1.6180$.

Example 4.1.7 ([Hare, Morris, Sidorov and Theys, 2011])**.** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \alpha \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ with

$$\alpha = 0.749326546330367557943961948091344672091327370236064317358024\ldots\ .$$

The set $\{A, B\}$ does not possess an s.m.p.. It is unknown whether α is rational or irrational. △

It is an open question whether pairs of binary matrices always posses an s.m.p. [Blondel and Jungers, 2008].

There are three common strategies to exploit (4.1.4);

(*i*) compute all products up to a length $k \in \mathbb{N}$ [Gripenberg, 1996; Moision, Orlitsky and Siegel, 2001; Möller and Reif, 2014];

(*ii*) take a suitable family of norms and minimize the right hand side of (4.1.4) with respect to this family [Blondel, Nesterov and Theys, 2005; Parrilo and Jadbabaie, 2008; Blondel, Jungers and Protasov, 2010; Ahmadi, Jungers, Parrilo and Roozbehani, 2011];

(*iii*) construct a norm which gives tight estimates in (4.1.4) for short products, preferably for products of length 1 [Guglielmi and Zennaro, 2008; Kozyakin, 2010a,b; Guglielmi and Protasov, 2013, 2016].

Blondel and Chang [2013] wrote a survey about the JSR computation, including most of the aforementioned algorithms and compared their performance to each other. The reader is referred to this work, if she aims for more informations. Here, we only mention a few algorithms. The *Gripenberg algorithm* [Gripenberg, 1996] was one of the first algorithms which gave good estimates for the JSR for a large number of matrix families in reasonable time. It works well up to a relative accuracy of 95%. It will be discussed in more detail in Section 4.3. The algorithms by Parrilo and Jadbabaie [2008] and Ahmadi, Jungers, Parrilo and Roozbehani [2011] construct unit balls which are the level sets of strictly positive, homogeneous, polynomials of even degree. These algorithms are mostly very fast up to a relative accuracy of 99%. All of these algorithms only approximate the JSR and so far, to the authors knowledge, there exist only two algorithms which can compute the exact value of the JSR for a large class of matrices: the tree-based branch and bound approach by Möller and Reif [2014] and the invariant polytope algorithm by Guglielmi and Protasov [2013, 2016]. In this paper we concentrate on the invariant polytope algorithm, and thus follow strategy (*iii*).

4.2 Invariant polytope algorithm

In this section we present the concepts used for the invariant polytope algorithm [Guglielmi and Protasov, 2013, 2016]. We also present the invariant polytope algorithm in 4.2.14. As already mentioned, the main idea is the construction of a suitable norm.

Definition 4.2.1. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$. An *extremal norm* (for $\mathcal{A}$) is any norm $||\cdot||$ which satisfies

$$||A_j x|| \leq \mathrm{JSR}(\mathcal{A}) \cdot ||x|| \text{ for all } x \in \mathbb{R}^s,\ A_j \in \mathcal{A}.$$

We next discuss the existence of extremal norms.

Definition 4.2.2. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$. $\mathcal{A}$ is called *irreducible* if the matrices $A \in \mathcal{A}$ do not posses any non-trivial common invariant subspace.

A set $\{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$ is irreducible if and only if there exists an invertible matrix B such that $B^{-1}A_jB$ has block upper triangular form for all $j \in \{1, \dots, J\}$ [Jungers, 2009, Section 2.1.1].

Barabanov [1988] showed that every irreducible family of matrices possesses an extremal norm. Example 4.2.3 shows that irreducibility is indeed crucial for the existence of an extremal norm.

Example 4.2.3 ([Jungers, 2009, Example 2.1])**.** Let $\mathcal{A} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$. The JSR of this set is clearly the spectral radius of the matrix, and thus is one. This set does not posses an extremal norm. Assume to the contrary there exists a vector norm $||\cdot||$ such that for the induced matrix norm $\left|\left|\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right|\right| = 1$. By sub-multiplicativity it follows that for all $k \in \mathbb{N}$

$$\left|\left|\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\right|\right| = \left|\left|\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k\right|\right| \leq \left|\left|\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right|\right|^k \leq 1.$$

This is clearly impossible due to the equivalence of norms in $\mathbb{R}^2$.

On the other hand, irreducibility is not necessary for a set of matrices to have an extremal norm. Let $\mathcal{A} = \{A\}$, $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then, the 1-norm is an extremal norm for $\mathcal{A}$. Indeed, $||A||_1 = 1 = \mathrm{JSR}(\mathcal{A})$. △

A simple construction of an extremal norm is given in Theorem 4.2.4.

Theorem 4.2.4 ([Berger and Wang, 1992; Guglielmi and Zennaro, 2008])**.** *Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$ be irreducible and for $v \in \mathbb{R}^s$ define*

$$P(v) = \mathrm{co} \bigcup_{\substack{n\in\mathbb{N}_0 \\ A_j\in\mathcal{A}}} \{\pm A_{j_n} \cdots A_{j_1} v\}. \tag{4.2.1}$$

(*i*) *If* $\mathrm{JSR}(\mathcal{A}) \geq 1$ *and for some* $v \in \mathbb{R}^s$ *the set* $P(v)$ *is bounded and has non-empty interior, then* $\mathrm{JSR}(\mathcal{A}) = 1$ *and* $P(v)$ *is the unit ball of an extremal norm* $||\cdot||$ *for* $\mathcal{A}$.

(*ii*) *Conversely, if* $\mathrm{JSR}(\mathcal{A}) = 1$, *then* $P(v)$ *is a bounded subset of* $\mathbb{R}^s$ *for any* $v \in \mathbb{R}^s$.

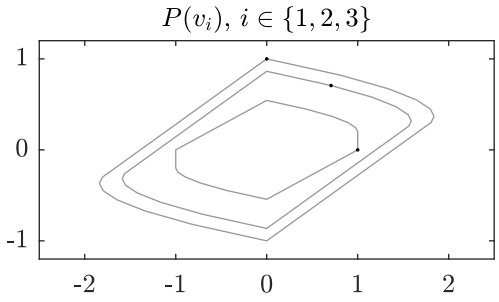


Figure 4.2: The sets $P(v_i)$, $i \in \{1,2,3\}$, for the matrices $\mathcal{C}_5 = \left\{ \frac{1}{e^{1/5}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \frac{e}{5} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ and vectors $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ as defined in Example 4.2.5.

Example 4.2.5. Let $\mathcal{C}_5 = \{C_0, C_5\}$ with $C_0 = \frac{1}{e^{1/5}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $C_5 = \frac{e}{5} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. We prove in Example 4.5.5 that $\mathrm{JSR}(\mathcal{C}_5) = 1$. The sets $P(v_i)$, $i \in \{1,2,3\}$ for the vectors $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $v_3 = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ are plotted in Figure 4.2. △

Definition 4.2.6. Let $P \subseteq \mathbb{C}^s$ be a compact, convex and *balanced* (i.e. $rP \subseteq P$ for all $|r| < 1$) set with non-empty interior. We define the *Minkowski norm* $||\cdot||_P : \mathbb{C}^s \to \mathbb{R}$ by

$$||x||_P = \inf \{r > 0 : x \in rP\}.$$

The Minkowski norm $||\cdot||_P$ is indeed a norm. It is finite because the interior of P is non-empty, positive definite because P is bounded and $0 \in P$, absolutely homogeneous because P is balanced and it satisfies the triangle inequality because P is convex.

The idea of the invariant polytope algorithm is to construct the set in (4.2.1) in finitely many steps, whenever it is a polytope. We describe polytopes by the convex hull of its vertices. The terminology in Definition 4.2.7 is adopted from [Guglielmi and Protasov, 2013].

Definition 4.2.7.

(*i*) For finite $V \subseteq \mathbb{R}^s$, we define the *convex hull* of V by

$$\mathrm{co}(V) = \left\{ x \in \mathbb{R}^s : x = \sum_{v \in V} t_v v \quad \text{with} \quad \sum_{v \in V} t_v \leq 1,\ t_v \geq 0 \right\}. \tag{4.2.2}$$

(*ii*) For finite $V \subseteq \mathbb{R}^s$, we define the *symmetrized convex hull* of V by

$$\mathrm{co}_s V = \left\{ x \in \mathbb{R}^s : x = \sum_{v \in V} t_v v \quad \text{with} \quad \sum_{v \in V} |t_v| \leq 1,\ t_v \in \mathbb{R}^s \right\} = \mathrm{co}(V \cup -V). \tag{4.2.3}$$

(*iii*) For finite $V \subseteq \mathbb{C}^s$, we define the *complex convex hull* or *absolutely convex hull* of V by

$$\mathrm{absco}\, V = \left\{ x \in \mathbb{C}^s : x = \sum_{v \in V} t_v v \quad \text{with} \quad \sum_{v \in V} |t_v| \leq 1,\ t_v \in \mathbb{C}^s \right\}. \tag{4.2.4}$$

(*iv*) For finite $V \subseteq \mathbb{R}^s_+$, we define the *cone hull* of V (with respect to the first orthant) by

$$\mathrm{co}_-\, V = \left\{ x \in \mathbb{R}^s_+ : x = y - z,\ y \in \mathrm{co}(V),\ z \in \mathbb{R}^s_+ \right\}. \tag{4.2.5}$$

(v) For simplicity, we denote with $\mathrm{co}_* V$ any of these convex hulls (co, co_s, absco, co_-) depending on the context.

In all cases we identify the (finite) set V with the matrix whose columns are the coordinates of the points $v \in V$.

Example 4.2.8. Let $V = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $x = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix}$. The norm of x corresponding to the symmetrized convex hull of V, $\mathrm{co}_s V$, defined by (4.2.3), computes to $||x||_{\mathrm{co}_s V} = \frac{9}{8}$. The norm of x corresponding to the cone hull of V, $\mathrm{co}_- V$, defined by (4.2.5), computes to $||x||_{\mathrm{co}_- V} = \frac{3}{4}$. Since the point x and all vertices $v \in V$ are purely real, the norm of x corresponding to the absolutely convex hull of V, $\mathrm{absco}\, V$, defined by (4.2.4), computes to $||x||_{\mathrm{absco}\, V} = ||x||_{\mathrm{co}_s V} = \frac{9}{8}$.

To compute the norms, one can use the LP-problems in Remark 4.2.9, or by using a ruler and the plot of the polytopes $\mathrm{co}_s V$ and $\mathrm{co}_- V$ given in Figure 4.3. The convex hull $\mathrm{absco}\, V$ is four-dimensional, and thus not plotted. The cone hull $\mathrm{co}_- V$ is only defined for the first orthant, and thus only plotted in the first orthant.

△

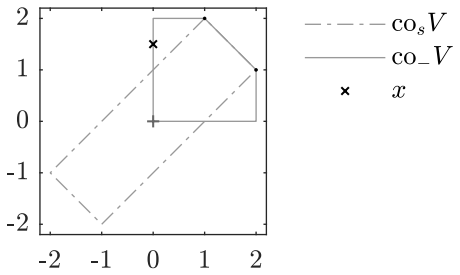


Figure 4.3: The polytopes $\mathrm{co}_s V$ and $\mathrm{co}_- V$ in grey, together with the point $x \in \mathbb{R}^2$ as defined in Example 4.2.8.

Remark 4.2.9. The Minkowski norm of a point with respect to the convex hull co_- or co_s of a finite set of vertices can be efficiently computed with linear programming [Guglielmi and Protasov, 2013, pages 11 and 21]. If $x \in \mathbb{R}^s$ and $V \subseteq \mathbb{R}^s$, finite, then

$$||x||_{\mathrm{co}_s}^{-1} = \begin{cases} \max t_0 \in \mathbb{R} & \\ \text{subject to} & t_0 x = \sum_{v \in V} t_v v \\ & -q_v \leq t_v \leq q_v, \quad q_v \geq 0 \quad \text{for all} \quad v \in V \\ & \sum_{v \in V} q_v \leq 1 \end{cases}. \tag{4.2.6}$$

If $x \in \mathbb{R}^s_+$ and $V \subseteq \mathbb{R}^s_+$, finite, then

$$||x||_{\mathrm{co}_-}^{-1} = \begin{cases} \max t_0 \in \mathbb{R} & \\ \text{subject to} & t_0 x \leq \sum_{v \in V} t_v v, \quad t_v \geq 0 \quad \text{for all} \quad v \in V \\ & \sum_{v \in V} t_v \leq 1 \end{cases}. \tag{4.2.7}$$

The function `computepolytopenorm` [Mejstrik, 2018b] computes the Minkowski norm of a point with respect to the convex hulls co_s or co_- of a set of vertices.

The Minkowski norm of a point with respect to the absolutely convex hull of a finite set of vertices can be computed using semi-definite programming [Guglielmi and Protasov, 2013, page 17]. If $x \in \mathbb{C}^s$ and $V \subseteq \mathbb{C}^s$, finite, then

$$||x||^{-1}_{\operatorname{absco} V} = \begin{cases} \max t_0 \in \mathbb{R} & \\ \text{subject to} & \sum_{v\in V} \Re(t_v)\Re(v) - \Im(t_v)\Im(v) = t_0\Re(x) \\ & \sum_{v\in V} \Re(t_v)\Im(v) + \Im(t_v)\Re(v) = t_0\Im(x) \quad \text{and} \\ & \sum_{v\in V} \sqrt{\Re(t_v)^2 + \Im(t_v)^2} \leq 1 \end{cases} \tag{4.2.8}$$

If $x \in \mathbb{R}^s$ and $V \subseteq \mathbb{R}^s$, then the computation of the norm in (4.2.8) with respect to the complex convex hull reduces to the computation of the norm with respect to the symmetrized convex hull, i.e. $||x||_{\operatorname{cos} V} = ||x||_{\operatorname{absco} V}$. The Minkowski-norm corresponding to the absolute convex hull of a finite set of vertices is not yet implemented in the function `computepolytopenorm`; it is intended to include it in a future release.

As one can see in Figure 4.2, the sets $P(v)$ defined in (4.2.1) are usually not polytopes. The idea of the invariant polytope algorithm lies in choosing a suitable vector $v \in \mathbb{R}^s$ for the set $P(v)$. More precisely, we will choose eigenvectors corresponding to the largest eigenvalues in modulus of certain matrix products.

Definition 4.2.10. Let $A \in \mathbb{R}^{s\times s}$.

(*i*) All eigenvalues of A equal to $\rho(A)$ in modulus are called *leading eigenvalues*.

(*ii*) All eigenvectors corresponding to leading eigenvalues are called *leading eigenvectors*.

(*iii*) An eigenvalue is *simple* if its (algebraic) multiplicity equals to 1.

(*iv*) A leading eigenvalue is *unique*, if there exists only 1 leading eigenvalue.

Example 4.2.11. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ has two simple eigenvalues 1 and -1; neither of them is unique since $|1| = |-1|$. △

We make use of the existence of a non-negative leading eigenvector in a special case in the invariant polytope algorithm 4.2.14.

Lemma 4.2.12 (Perron-Frobenius theorem)**.** *If $A = [a_{i,j}]_{i,j=1}^s \in \mathbb{R}^{s\times s}$, $a_{i,j} \geq 0$ for all $i, j \in \{1, \ldots, s\}$, then there exists a non-negative leading eigenvalue and the corresponding leading eigenvector can be chosen such that all of its entries are non-negative.*

Next we present the invariant polytope algorithm 4.2.14 in a simplified form. In this form the algorithm will presumably not terminate. However, the core idea of the algorithm should become clear. Necessary conditions for termination of the algorithm are given in Theorem 4.2.20. The full algorithm is given, together with our proposed modifications, in Section 4.4.

Definition 4.2.13. Let $A \in \mathbb{R}^{s\times s}$, $v \in \mathbb{R}^s$, $w = Av$. We call the vertex w a *child* (or an *alma*) of v, and v a *parent* of w.

Algorithm and Definition 4.2.14 (Invariant polytope algorithm [Guglielmi and Protasov, 2013, 2016])**.** Given a finite set of matrices $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$.

(1) For some $D \in \mathbb{N}$ look over all products of matrices in $\mathcal{A}$ of length less than D and choose any shortest product Π_1 such that $\rho_c \coloneqq \rho(\Pi_1)^{1/l(1)}$ is maximal, where $l(1)$ is the length of the product Π_1 and call Π_1 *spectral maximizing product-candidate* (*s.m.p.-candidate*). Set $\tilde{\Pi}_1 \coloneqq \rho_c^{-l(1)}\Pi_1$, $\tilde{A}_j = \rho_c^{-1}A_j$, $j \in \{1, \dots, J\}$, $\tilde{\mathcal{A}} = \left\{\tilde{A}_j : j = 1, \dots, J\right\}$.

Next we try to prove that $\mathrm{JSR}(\tilde{\mathcal{A}}) \leq 1$.

(2) Choose a leading eigenvector $v_1^{(0)} \in \mathbb{C}^s$ such that $\tilde{\Pi}_1 v_1^{(0)} = \lambda v_1^{(0)}$, $|\lambda| = 1$.

(3) If $l(1) \geq 2$, then we compute the corresponding leading eigenvectors of the cyclic permutations of $\tilde{\Pi}_1$, i.e. $v_1^{(i)} \coloneqq \tilde{A}_{j_i} \cdots \tilde{A}_{j_1} v_1$, $i \in \{1, \dots, l(1) - 1\}$.

We define the *cyclic root* $\mathcal{H} \coloneqq \left\{v_1^{(0)}, \dots, v_1^{(l(1)-1)}\right\}$ and set $V \coloneqq \mathcal{H}$.

(4) For all $v \in \mathcal{A}V \setminus V$ do

If $||v||_{\mathrm{co}_* V} > 1$, then set $V \coloneqq V \cup v$.
Depending on the matrices in $\mathcal{A}$ and the leading eigenvector $v_1^{(0)}$ we use different convex hulls:

- *Case* (P): If all entries of the matrices A_j, $j \in \{1, \dots, J\}$, are non-negative, then we take a non-negative leading eigenvector v in step (2) and use co_-.
- *Case* (R): If the matrices A_j, $j \in \{1, \dots, J\}$, have positive and negative entries and the chosen leading eigenvector $v_1^{(0)}$ is real, then we use co_s.
- *Case* (C): If the matrices A_j, $j \in \{1, \dots, J\}$, have positive and negative entries and the chosen leading eigenvector $v_1^{(0)}$ is complex, then we use absco.

(5) Repeat step (4) until $\tilde{\mathcal{A}}V \subseteq \mathrm{co}_* V$.

(6) If the loop terminates, then the algorithm terminates and we found the invariant polytope V, which implies that $\left|\left|\tilde{A}_j\right|\right|_{\mathrm{co}_* V} \leq 1$ for all $j \in \{1, \dots, J\}$; in other words $\mathrm{JSR}(\tilde{\mathcal{A}}) \leq 1$.

Remark 4.2.15. In step 4.2.14 (4), we actually add a vertex $v \in \tilde{\mathcal{A}}V \setminus V$ even if it lies slightly inside of the polytope, i.e. whenever $||v||_{\mathrm{co}_* V} > 1 - \epsilon$, where ϵ is the tolerance within which we can compute the norm.

Example 4.2.16. Let $\mathcal{A} = \{A, B\}$ with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. All entries of A and B are non-negative, and thus we are in case (P) and use the cone hull co_- to compute the Minkowski-norms in step 4.2.14 (4).

(1) We choose

$$\Pi_1 = AAB = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix},$$

which is the product with the highest averaged spectral radius among all products of length less or equal than three. Thus, $l(1) = 3$, $\rho_c = \rho(\Pi_1)^{1/l(1)} = 3^{1/3}$ and we define $\tilde{A} = \rho_c^{-1}A$, $\tilde{B} = \rho_c^{-1}B$, $\tilde{\mathcal{A}} = \left\{\tilde{A}, \tilde{B}\right\}$, $\tilde{\Pi}_1 = \tilde{A}\tilde{A}\tilde{B}$.

(2) The s.m.p.-candidate $\tilde{\Pi}_1$ has a unique and simple leading eigenvalue 1 with a corresponding eigenvector $v_1 = v_1^{(0)}$ given by

$$v_1 = v_1^{(0)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(3) We construct the cyclic root

$$\mathcal{H} = \left\{ v_1^{(0)}, v_1^{(1)}, v_1^{(2)} \right\} = \left\{ v_1, \tilde{B}v_1, \tilde{A}\tilde{B}v_1 \right\} = \frac{1}{\sqrt{5}} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3^{2/3} \end{bmatrix}, \begin{bmatrix} 3^{1/3} \\ 3^{1/3} \end{bmatrix} \right\}$$

and set $V = \mathcal{H}$.

(4) We compute the norms of the vectors $\tilde{\mathcal{A}}V \setminus V$. The vector $\tilde{A}v_1$ is outside of the polytope $\operatorname{co}_- V$,

$$\|\tilde{A}v_1\|_{\operatorname{co}_- V} \simeq 1.04 \geq 1,$$

and thus it is added to the set V. All other vectors, i.e. $\tilde{B}\tilde{B}v_1$ and $\tilde{B}\tilde{A}\tilde{B}v_1$, in the first iteration are inside of $\operatorname{co}_- V \cup \tilde{A}v_1$;

$$\|\tilde{B}\tilde{B}v_1\|_{\operatorname{co}_- V \cup \tilde{A}v_1} \simeq 0.69 < 1, \quad \|\tilde{B}\tilde{A}\tilde{B}v_1\|_{\operatorname{co}_- V \cup \tilde{A}v_1} \simeq 0.96 < 1.$$

We repeat step (4) and test the vectors from the set $\mathcal{A}(V \cup \tilde{A}v_1) \setminus (V \cup \tilde{A}v_1)$;

$$\|\tilde{A}\tilde{A}v_1\|_{\operatorname{co}_- V \cup \tilde{A}v_1} \simeq 0.92 < 1, \quad \|\tilde{B}\tilde{A}v_1\|_{\operatorname{co}_- V \cup \tilde{A}v_1} \simeq 0.92 < 1.$$

(5) All vertices from the last iteration are mapped into the interior of the polytope $P = \operatorname{co}_- V \cup \tilde{A}v_1$, and thus P is $\tilde{\mathcal{A}}$-invariant and $\operatorname{JSR}(\mathcal{A}) = \rho(\Pi_1)^{1/l(1)} = 3^{1/3} \simeq 1.4422$.

The computed points and the cone hull of the computed points are printed in Figure 4.4. △

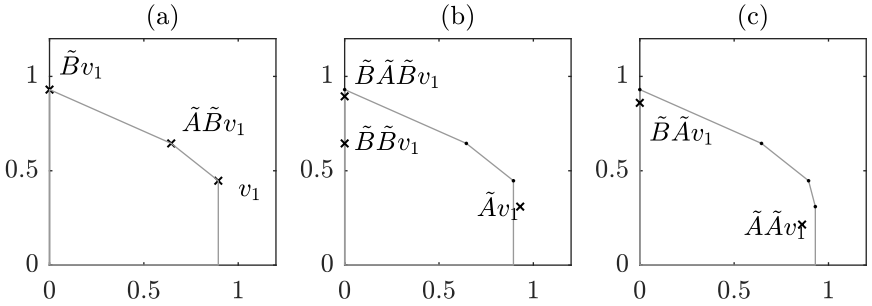


Figure 4.4: The polytope $\operatorname{co}_- V$ as constructed by the invariant polytope algorithm 4.2.14 for the matrices defined in Example 4.2.16. In (a) we see the cone with respect to the cyclic root $\mathcal{H} = \{v_1, \tilde{B}v_1, \tilde{A}\tilde{B}v_1\}$. In (b) we see the vertices $\tilde{A}v_1$, $\tilde{B}\tilde{B}v_1$ and $\tilde{B}\tilde{A}\tilde{B}v_1$ constructed in the first iteration. In (c) we see the new polytope $\mathcal{H} \cup \tilde{A}v_1$ together with the vertices $\tilde{A}\tilde{A}v_1$ and $\tilde{B}\tilde{A}v_1$ constructed in the second iteration, which are all mapped into the interior of $\operatorname{co}_- \mathcal{H} \cup \tilde{A}v_1$.

If there is more than one s.m.p. in step 4.2.14 (1), Guglielmi and Protasov [2016] showed that one has to construct multiple cyclic trees $\mathcal{H}_r$, $r \in \{1, \ldots, R\}$, in step 4.2.14 (3) and balance the sizes of the corresponding convex hulls to each other. This is done using the dual leading eigenvectors which we define next.

Definition 4.2.17 ([Guglielmi and Protasov, 2016, Section 2.3])**.** Let $A \in \mathbb{R}^{s \times s}$ and $v \in \mathbb{R}^s$ be a leading eigenvector corresponding to the unique and simple leading eigenvalue λ of A. We define the *dual leading eigenvector* v^* (corresponding to v) by

$$A^* v^* = \lambda v^* \quad \text{with} \quad (v, v^*) = 1.$$

The existence of v^* in 4.2.17 follows from a standard argument.

Lemma 4.2.18. *Whenever $A \in \mathbb{R}^{s\times s}$ has a unique and simple leading eigenvalue λ with corresponding eigenvector v, the dual leading eigenvector v^* as defined in 4.2.17 exists.*

Proof. Since $\det(A-\lambda I) = \det(A^*-\lambda I)$, the left and right-eigenvalues of A, as well as their multiplicities, are the same. Thus, there exists, up to normalization, a unique simple left-eigenvector v^* to λ. The fact that λ is simple, implies that there exists an invertible matrix Λ such that

$$A = \Lambda \begin{bmatrix} \lambda & 0 \\ 0 & \tilde{A} \end{bmatrix} \Lambda^{-1}, \quad \text{with } \tilde{A} \in \mathbb{R}^{(s-1)\times(s-1)}.$$

The first column of Λ^{-1} is a right eigenvector of A corresponding to λ and the first row of Λ is a left eigenvector of A corresponding to λ. Since $\Lambda\Lambda^{-1} = I$, we can choose v^* such that $(v, v^*) = 1$. □

We next describe the balancing procedure. Assume we found s.m.p.-candidates $\Pi_1, \dots, \Pi_R$, $R \in \mathbb{N}$. Then, we need to find factors $\alpha_q > 0$ such that, for some $h \in \mathbb{N}$,

$$\alpha_q \max_{z\in\tilde{\mathcal{A}}^h \left\{v_q^{(0)},\dots,v_q^{(l(q)-1)}\right\}} |(v_r^*, z)| < \alpha_r, \quad \text{for all} \quad q, r \in \{1, \dots, R\}.$$

Afterwards set $\mathcal{H}_q \coloneqq \left\{\alpha_q v_q^{(0)}, \dots, \alpha_q v_q^{(l(q)-1)}\right\}$, $v_q^* \coloneqq \alpha_q^{-1} v_q^*$, $q \in \{1, \dots, R\}$, and $V \coloneqq \bigcup_{q=1}^R \mathcal{H}_q$.

Example 4.4.4 gives a set of matrices, for which it was wrongly assumed that all balancing factors are equal to one.

The conditions in Theorem 4.2.20 under which the invariant polytope algorithm 4.2.14 terminates sound very strong, but all numerical examples suggest that they are rather general.

Definition 4.2.19. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$. An s.m.p. Π_1 is called *dominant* if there exists $\gamma > 0$ such that $\rho(A_{j_m} \cdots A_{j_1})^{1/m} < \gamma < \mathrm{JSR}(\mathcal{A}) = \rho(\Pi_1)^{1/l(1)}$, $A_j \in \mathcal{A}$, whenever $A_{j_m} \cdots A_{j_1}$, $A_j \in \mathcal{A}$, is not an s.m.p..

Theorem 4.2.20 ([Guglielmi and Protasov, 2016, Theorem 3.3])**.** *Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$. The invariant polytope algorithm, for cases (P) and (R), terminates and returns* $\mathrm{JSR}(\mathcal{A})$ *if and only if*

(i) there exist only finitely many s.m.p.s Π_r, $r \in \{1, \dots, R\}$, apart from their cyclic permutations and powers,

(ii) the s.m.p.s Π_r, $r \in \{1, \dots, R\}$, are dominant and

(iii) the leading eigenvectors $v_r^{(0)}$ of Π_r, $r \in \{1, \dots, R\}$, are unique and simple.

As far as we know, the invariant polytope algorithm 4.2.14 offers the only characterization, whether a set of matrices has dominant s.m.p.s or not.

Remark 4.2.21. The conditions in Theorem 4.2.20 implicitly exclude the case of matrices with zero joint spectral radius. Indeed, if $\mathrm{JSR}(\mathcal{A}) = 0$, then every matrix product of matrices in $\mathcal{A}$ has spectral radius zero, and thus there are infinitely many s.m.p.s. Therefore, we may always assume that $\rho_c > 0$, and thus we can define the set $\tilde{\mathcal{A}} = \rho_c^{-1}\mathcal{A}$ in the invariant polytope algorithm in step 4.2.14 (1).

Note that there is a polynomial time algorithm by Gurvits [1995] to decide whether the joint spectral radius of a set of matrices is zero or not, with an implementation by Hendrickx, Jungers and Vankeerberghen [2011].

There is a subtle difference in the existence of dominant s.m.p.s in dimension $s = 1$ and $s \geq 2$. Our numerical examples indicate, that most invariant sets of matrices possess a dominant s.m.p.. However, there exist sets of matrices without any s.m.p., see Example 4.1.7. On the contrary, in dimension $s = 1$, there always exist an s.m.p., and (in general) all s.m.p.s are not dominant. Indeed, if $a, b \in \mathbb{C}$, $|a| > |b|$, then $\mathrm{JSR}(\{a, b\}) = |a|$, and thus a is an s.m.p., but a is not dominant, since $\rho(a^n b)^{1/n} \to |a|$ as $n \to \infty$.

4.2.1 Summary of the main modifications of the invariant polytope algorithm

In this section we present the modifications of the invariant polytope algorithm 4.2.14 and explain their importance. The modifications are explained in detail in Sections 4.3 and 4.4.

4.2.2 Nearly s.m.p.s

Definition 4.2.22 ([Guglielmi and Protasov, 2016, Remark 3.7])**.** Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \ldots, J\}$. Any matrix product whose averaged spectral radius nearly equals the joint spectral radius of $\mathcal{A}$ is called *nearly-s.m.p.*.

If the spectral gap between the averaged spectral radii of the nearly-s.m.p.s and the JSR is small, then the generated polytope very slowly absorbs new vertices. In this case, it is helpful to consider also the polytopes generated by the *nearly-s.m.p.s*. The suggested balancing procedure of the nearly-s.m.p.s in [Guglielmi and Protasov, 2016, Remark 3.7] does not always work, as Example 4.4.5 shows. Section 4.4.5 presents a new method to circumvent this problem.

4.2.3 Extra-vertices

Definition 4.2.23 ([Guglielmi and Protasov, 2016, Section 4])**.** Every vertex $v \in \mathbb{R}^s$, added to the set V in step 4.2.14 (3), which is not a leading eigenvector corresponding to an s.m.p.-candidate or a nearly-s.m.p. is called *extra vertex*.

The concept of extra vertices can speed up the invariant polytope algorithm 4.2.14 in cases where the constructed polytope is very flat. In Section 4.4.4 we present a method which automatically chooses an appropriate set of extra-vertices. This task was done by hand so far.

4.2.4 Finding s.m.p. candidates

The invariant polytope algorithm 4.2.14 only terminates, if the s.m.p.-candidates Π_r, $r \in \{1, \ldots, R\}$ are indeed s.m.p.s.. Thus, the invariant polytope algorithm 4.2.14 heavily relies on a correct initial guess for the s.m.p.-candidates. A plain brute-force search in step 4.2.14 (1) will fail, if the s.m.p.s' length is large. Our numerical tests have shown that even for random pairs of matrices, s.m.p.s of length greater than 30 are not uncommon. A particularly easy example of two matrices with a very long s.m.p. is given in Example 4.5.5. We present two new efficient methods that search for s.m.p.s in Sections 4.3 and 4.4.11.

4.2.5 Bounds for the JSR

Whenever the invariant polytope algorithm 4.2.14 does not find an invariant polytope in reasonable time, it is claimed that the original invariant polytope algorithm 4.2.14 returns upper bounds for the joint spectral radius if one keeps track of the norms $||v||_{\mathrm{co}_* V}$, $v \in \tilde{\mathcal{A}}V \setminus V$, in 4.2.14 (4) [Guglielmi and Protasov, 2013, Section 2.1]. However, their argument contains a gap, and the returned bounds by the original algorithm are not sure to be correct in all cases. We show in Section 4.4.10 how this can happen.

With our modification in Section 4.4.10 the modified invariant polytope algorithm 4.4.1 returns intermediate bounds for the JSR and we prove that they are correct. Nevertheless, these bounds are usually quite rough. A simple modification, presented in Remark 4.4.3, increases the accuracy of these intermediate bounds. However, the drawback of the modification in Remark 4.4.3 is that the exact value of the JSR becomes uncomputable.

4.2.6 Parallelisation and natural selection of vertices

A disadvantage of the invariant polytope algorithm 4.2.14 in its current form is that the polytope is changed *inside* of the main loop in 4.2.14 (4), which implies that the norm of $v \in \tilde{\mathcal{A}}V \setminus V$, in 4.2.14 (4) has to be computed with respect to a different polytope for each vertex $v \in \tilde{\mathcal{A}}V \setminus V$. Therefore, the linear programming problem for computing the norm is different for each norm and the so-called warm start of linear programming problems cannot be used. Furthermore, the main loop 4.2.14 (4) cannot be parallelised. We eliminate these two problems and additionally speed up the algorithm in Section 4.4.8.

With the same technique we can solve problems arising when the number of matrices in $\mathcal{A}$ is large. In such cases algorithm 4.2.14 stalls simply due to the fact that the number of vertices to test increases (in the worst case) by a factor of $\#\mathcal{A}$ in each iteration. For example, if $\#\mathcal{A} = 256$, as in one test case in Section 4.5.3, the original invariant polytope algorithm 4.2.14 will never reach the third iteration.

4.2.7 Estimating the Minkowski norm

To reduce the number of norms one has to compute in step 4.2.14 (4), and thus to speed up the algorithm, we use the estimates for the Minkowski norm in Lemma 4.4.6.

4.3 Modified Gripenberg algorithm

In this section, we present the modified Gripenberg algorithm.

From (4.1.4) we know that the averaged spectral radius of any matrix product is a lower bound for the JSR. Thus, by a clever guess of a matrix product one easily obtains good (maybe even sharp) lower bounds for the JSR. The modified Gripenberg algorithm 4.3.5 finds in nearly all of our numerical tests an s.m.p.. It is a modification of the well known *Gripenberg algorithm* [Gripenberg, 1996]; one of the first algorithms which gave reasonable estimates for the JSR. We will describe the Gripenberg algorithm before we present the proposed modifications.

4.3.1 Gripenberg algorithm

Given a finite set of matrices $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s}, j = 1, \dots, J\}$, the Gripenberg algorithm sorts out matrix products, which are proven to have averaged spectral radius less than the joint spectral radius of the given set. In this way, the number of matrix products to be computed are kept reasonably small.

Precisely, given some accuracy $0 < \delta \leq 1$, we iteratively compute the sets $C_k \subseteq \mathbb{R}^{s\times s}$, $k \in \mathbb{N}$, by $C_1 := \mathcal{A}$ and C_{k+1} consists of all matrices $C \in \mathcal{A}C_k$ with $||C||^{1/(k+1)} \geq \delta^{-1} b_-$, where $b_- = \max_{n\in\{1,\dots,k\}}\{\rho(C)^{1/n} : C \in C_n\}$ is the current lower bound for the JSR. For each k the JSR lies in the interval $[b_-, b_+]$ with $b_+ = \min_{n\in\{1,\dots k\}} \max\{||C||^{1/n} : C \in C_n\}$.

Theorem 4.3.1 ([Gripenberg, 1996]). *Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$. Whenever $\delta < 1$, the Gripenberg algorithm terminates and returns $b_-, b_+ \geq 0$ such that $\mathrm{JSR}(\mathcal{A}) \in [b_-, b_+]$ and $b_-/b_+ \leq \delta$.*

For real-world applications the Gripenberg algorithm works well for $\delta \leq 0.95$. For larger δ the number of products to compute gets usually too large.

4.3.2 Modified Gripenberg algorithm

The modified Gripenberg algorithm 4.3.5 uses a different selection mechanism. Instead of dismissing products with norms less than some threshold, it keeps the products with highest and lowest norms; all products whose norms lie in between are disregarded. This way, the modified Gripenberg algorithm 4.3.5 cannot determine upper bounds for the JSR anymore, however it still works for finding s.m.p.s of considerable length. The modified Gripenberg algorithm 4.3.5 is listed on page 73.

The modified Gripenberg algorithm 4.3.5 depends on two parameters N and D, controlling the number of products kept in each iteration and the maximum length of the considered products, respectively. Note that the modified Gripenberg algorithm 4.3.5 with parameters $N = D = \infty$ is the Gripenberg algorithm with accuracy $\delta = 1$.

Theorem 4.3.2. *The modified Gripenberg algorithm 4.3.5 has linear complexity in the number $J = \#\mathcal{A}$ of matrices, in the number $N \in \mathbb{N}$ of kept products in each iteration and in the maximal length $D \in \mathbb{N}$ of the products.*

Proof. In every iteration, in total D many, the modified Gripenberg algorithm 4.3.5 computes at most $2 \cdot N \cdot J$ norms and spectral radii. □

Remark 4.3.3. Clearly one can pursue other selection strategies in the modified Gripenberg algorithm 4.3.5 in step (4.3.2). The straightforward choice of taking the $2 \cdot N$ products with highest averaged norm performs very badly.

Taking an arbitrary subset of $\mathcal{M}_d$ of size $2 \cdot N$ performs mostly similarly to the proposed modified Gripenberg algorithm 4.3.5, yet in some cases worse, see Figure 4.9. Furthermore, the modified Gripenberg algorithm 4.3.5 in the given form is deterministic, so we prefer it over a non-deterministic version.

Example 4.3.4. Let $\mathcal{A} = \{A, B\}$ with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $N = 1$, $D = 3$.

In the first iteration we compute $||A||_2 \simeq 1.61$, $||B||_2 \simeq 1.41$, $\rho(A) = 1$, $\rho(B) = 1$. Thus, $\rho_c = 1$. Since we keep $2 \cdot N$ matrix products in each step, we do not sort out any matrix product.

In the second iteration we compute

$$\begin{aligned} &||AA||_2^{1/2} \simeq 1.55, && ||BA||_2^{1/2} \simeq 1.49, && ||AB||_2^{1/2} \simeq 1.41, && ||BB||_2^{1/2} \simeq 1.19, \\ &\rho(AA)^{1/2} = 1, && \rho(BA)^{1/2} \simeq 1.41, && \rho(AB)^{1/2} \simeq 1.41, && \rho(BB)^{1/2} = 1, \end{aligned}$$

and get a new lower bound $\rho_c \simeq 1.41$. Since $||BB||^{1/2} < \rho_c$, we disregard this product. From the remaining products we choose those with highest and lowest norm, i.e. AA and AB.

In the third iteration we compute the averaged norms and averaged spectral radii of the matrix products

$$\begin{aligned} &||AAA||_2^{1/3} \simeq 1.44, && ||BAA||_2^{1/3} \simeq 1.47, && ||AAB||_2^{1/3} \simeq 1.47, && ||BAB||_2^{1/3} \simeq 1.41, \\ &\rho(AAA)^{1/3} = 1, && \rho(BAA)^{1/3} \simeq 1.44, && \rho(AAB)^{1/3} \simeq 1.44, && \rho(BAB)^{1/3} \simeq 1.26, \end{aligned} \tag{4.3.1}$$

and get a new lower bound $\rho_c \simeq 1.44$. If we continue with this algorithm, we disregard the product BAB, since $||BAB||_2^{1/3} < \rho_c$ and choose the product AAA which has the smallest averaged norm among the remaining products. Moreover, we choose either BAA or AAB which have the largest averaged norm in (4.3.1). Since $D = 3$, the modified Gripenberg algorithm 4.3.5 returns $\rho_c \simeq 1.44$ as a lower bound for $\mathrm{JSR}(\mathcal{A})$ and the product AAB as an s.m.p.-candidate.

The products computed in this example, together with the values for the averaged norms and spectral radii, are plotted in Figure 4.5. △

Algorithm 4.3.5 (modified Gripenberg algorithm)**.**

Input :

Set of square matrices $\mathcal{A} = \left\{A_j \in \mathbb{R}^{s\times s} : j = 1, \ldots, J\right\}$.

Number of products kept in each step $N \in \mathbb{N}$.

Maximal length of products $D \in \mathbb{N}$.

Output :

S.m.p.-candidates $\mathcal{C}$.

Lower bound ρ_c for JSR($\mathcal{A}$).

Initialization :

Start with the product of length 0, $\mathcal{M}_0 \coloneqq \{I\}$, where I is the identity matrix.

Set current lower bound for JSR, $\rho_c \coloneqq 0$.

Main Loop :

for $d \in \{1, \ldots, D\}$ **do**

Compute all possible new products $\mathcal{M}_d \coloneqq \mathcal{A}\mathcal{M}_{d-1}$.

Update the lower bound $\rho_c \coloneqq \max\left\{\rho_c, \rho(M_d)^{1/d} : M_d \in \mathcal{M}_d\right\}$.

Remove products whose norms are less than ρ_c, $\mathcal{M}_d \coloneqq \left\{M_d \in \mathcal{M}_d : ||M_d||^{1/d} \geq \rho_c\right\}$.

Keep only products with highest and lowest norms:

Sort $\mathcal{M}_d$ w.r.t $||M_d||$ and keep $M_1, \ldots, M_N$ and $M_{\#\mathcal{M}_d-N}, \ldots, M_{\#\mathcal{M}}$. (4.3.2)

Thus, $\mathcal{M}_d \coloneqq \{M_1, \ldots, M_N, M_{\#\mathcal{M}_d-N}, \ldots, M_{\#\mathcal{M}_d} : M_i \in \mathcal{M}_d\}$.

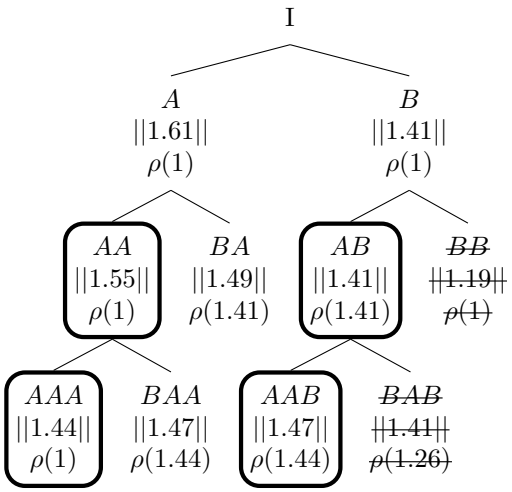
end

Choose products $\mathcal{C} \coloneqq \left\{M_{d_i} \in \mathcal{M}_d : \rho(M_{d_i})^{1/d} = \rho_c,\ d = 1, \ldots, D\right\}$.

Remove cyclic permutations and powers of products from $\mathcal{C}$.

return $\mathcal{C}$, ρ_c.

Figure 4.5: Tree built up by the modified Gripenberg algorithm 4.3.5, with parameter $N = 1$, $D = 3$, for the matrices defined in Example 4.3.4. The algorithm terminates after $D = 3$ steps and returns the product AAB as an s.m.p.-candidate. The values in $||\cdot||$ and $\rho(\cdot)$ denote the averaged norm and averaged spectral radius of the corresponding product.



Remark 4.3.6. The function `findsmp` [Mejstrik, 2018b] is an implementation of the modified Gripenberg algorithm 4.3.5. A simple implementation of the modified Gripenberg algorithm 4.3.5 for Matlab is listed on page 95. The spectral radii are computed with the Matlab function `eig`, which may not be the fastest available procedure.

The modified invariant polytope algorithm 4.4.1 can also be used to search for s.m.p.-candidates. Thus, we present the numerical examples showing the performance of the modified Gripenberg algorithm 4.3.5 only after Section 4.4.

4.4 Modified invariant polytope algorithm

In this section we present the modifications for the invariant polytope algorithm 4.2.14. Its current implementation [Mejstrik, 2018b, December 2018] only handles the case (P) (as explained in 4.2.14 (4)) with non-negative matrices and the case (R) with real leading eigenvalues. Case (C) with complex leading eigenvalues is not yet implemented; it is intended to include this case in a future release. Therefore, we restrict the discussion of the modified invariant polytope algorithm 4.4.1 to the cases (P) and (R). Nevertheless, most of the results are valid in the case (C) as well.

The modified invariant polytope algorithm 4.4.1 is listed on page 75. We prove the termination criteria and the correctness of the returned bounds in Theorem 4.4.2.

Theorem 4.4.2. *Let $\mathcal{A} = \{A_j : j = 1, \dots, J\} \subseteq \mathbb{R}^{s\times s}$ be a finite set of square matrices, $\delta \in (0, 1]$ and ρ_c be the averaged spectral radius of the s.m.p.-candidates Π_r, $r \in \{1, \dots, R\}$.*

(i) For $\delta = 1$, the modified invariant polytope algorithm 4.4.1 terminates, and thus $\mathrm{JSR}(\mathcal{A}) = \rho_c$, if and only if the original invariant polytope algorithm 4.2.14 terminates.

(ii) For $0 < \delta < 1$ the modified invariant polytope algorithm 4.4.1 terminates, and thus $\mathrm{JSR}(\mathcal{A}) \in [\rho_c,\ \delta^{-1}\rho_c]$, if $\mathrm{JSR}(\mathcal{A}) < \delta^{-1} \cdot \rho_c$.

(iii) Moreover, for any iteration $k \in \mathbb{N}_0$, $\mathrm{JSR}(\mathcal{A}) \in [\rho_c,\ \delta^{-1} \cdot b_{k+1} \cdot \rho_c]$, where b_{k+1} is defined in Section 4.4.10.

Before we present the proof of Theorem 4.4.2 on page 82, we describe all modifications and extensions of the modified invariant polytope algorithm 4.4.1. These are numbered (4.4.1)-(4.4.11) in the modified invariant polytope algorithm 4.4.1.

Remark 4.4.3. The function `tjsr` [Mejstrik, 2018b] is an implementation of the modified invariant polytope algorithm 4.4.1.

The most important input parameters of our implementation of the modified invariant polytope algorithm are described alongside the description of the modifications in Sections 4.4.1 to 4.4.11. A full list of the input parameters is given in the manual together with our implementation.

4.4.1 Irreducibility of input matrices (4.4.1)

The input matrices in $\mathcal{A}$ should be irreducible, i.e. do not have a trivial common invariant subspace. Otherwise, the modified invariant polytope algorithm 4.4.1 may not be able to terminate, since the existence of an extremal norm is not ensured, see Example 4.2.5. If the matrices in $\mathcal{A}$ are reducible, then there exists a basis in which all of the matrices A_j have block upper triangular form, see e.g. [Jungers, 2009, Proposition 1.5]. In this case, the JSR of $\mathcal{A}$

Algorithm 4.4.1 (Modified invariant polytope algorithm)**.** All modifications to algorithm 4.2.14 are marked with $*$ below. Numbered lines are subroutines described in detail in Sections 4.4.1 – 4.4.11.

Input :

Set of irreducible square matrices $\mathcal{A} = \left\{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\right\}$. (4.4.1)

Tolerance $0 < \epsilon < 1$ for computing the norms $N(v)$ in (4.4.8) ($\epsilon \simeq 0$).

Accuracy $0 < \delta \leq 1$ to which the JSR shall be computed ($\delta \simeq 1$).

Output :

Exact value ρ_c of JSR($\mathcal{A}$) or bound $[\rho_c, b \cdot \rho_c]$ for JSR($\mathcal{A}$).

Invariant polytope V.

Spectral maximizing products $\Pi_r,\ r \in \{1, \dots, R\}$.

Initialization :

$*$ Search for s.m.p.-cand. and nearly-s.m.p.s $\Pi_r = A_{j_{r_{l(r)}}} \cdots A_{j_{r_1}},\ r \in \{1, \dots, R\},\ R \in \mathbb{N}$,

and denote with $l(r) = l(\Pi_r)$ the lenght of the product Π_r. (4.4.2)

Set $\rho_r \coloneqq \rho(\Pi_r)^{1/l(r)},\ \rho_c \coloneqq \max \rho_r,\ \tilde{\mathcal{A}} \coloneqq \delta\rho_c^{-1}\mathcal{A},\ \tilde{\Pi}_r = (\delta\rho_c^{-1})^{l(r)}\Pi_r$. (4.4.3)

Compute the leading eigenvectors $v_r^{(0)}$ of $\tilde{\Pi}_r$.

For all $r \in \{1, \dots, R\}$ where $l(r) \geq 2$, we compute the root vectors

$v_r^{(i)} \coloneqq (\rho_c/\delta\rho_r)^i \tilde{A}_{j_{r_i}} \dots \tilde{A}_{j_{r_1}} v_r^{(0)},\ i \in \{1, \dots, l(r) - 1\}$.

$*$ Compute extra-vertices $v_{R+1}, \dots, v_Q \in \mathbb{R}^s,\ Q \in \mathbb{N}$. (4.4.4)

$*$ Provide the balancing factors $\alpha_1, \dots, \alpha_Q \in \mathbb{R}$. (4.4.5)

Set $\mathcal{H} \coloneqq \left\{\alpha_1 v_1^{(0)}, \dots, \alpha_1 v_1^{(l(1)-1)}, \dots, \alpha_R v_R^{(1)}, \dots, \alpha_R v_R^{(l(R)-1)}\right\}$.

Set $V_0 \coloneqq \mathcal{H} \cup \{\alpha_{R+1} v_{R+1}, \dots, \alpha_Q v_Q\}$.

Set $N(v) \coloneqq \infty$ for all $v \in V_0$, $\quad b_0 \coloneqq \infty$, $\quad k \coloneqq 0$.

Main Loop :

while $\tilde{\mathcal{A}}V_k \setminus V_k \not\subseteq (1 - \epsilon)\operatorname{co}_* V_k$

$*$ Select new children $E_{k+1} \subseteq \tilde{\mathcal{A}}V_k \setminus V_k$. (4.4.6)

$*$ Choose subset of vertices $W_k \subseteq V_k$. (4.4.7)

$*$ Compute/estimate norms $N(v) \coloneqq ||v||_{\operatorname{co}_* W_k}$ for all $v \in E_{k+1}$. (4.4.8)

$V_{k+1} \coloneqq V_k \cup \{v \in E_{k+1} : N(v) > 1 - \epsilon\}$. (4.4.9)

$*$ Compute bound $b_{k+1} \geq 1$ for JSR($\tilde{\mathcal{A}}$). (4.4.10)

$*$ Restart if a matrix product with higher averaged spectral radius is found. (4.4.11)

Print JSR($\mathcal{A}$) $\in [\rho_c,\ \delta^{-1} \cdot b_{k+1} \cdot \rho_c]$.

Increase $k \coloneqq k + 1$.

end

return $V,\ \{\Pi_i\}_i,\ \rho_c$.

equals to the maximum of the JSR of the corresponding diagonal blocks. In our implementation [Mejstrik, 2018b] we, therefore, automatically search for non-trivial common invariant subspaces prior to starting the algorithm. Here we make use of the functions `permTriangul` and `jointTriangul` from [Hendrickx, Jungers and Vankeerberghen, 2011] as well as a new method `invariantsubspace` [Mejstrik, 2018b] which searches for non-trivial common invariant difference subspaces as described in [Charina and Protasov, 2017].

4.4.2 Search for s.m.p.-candidates (4.4.2)

In our implementation [Mejstrik, 2018b] we use the modified Gripenberg algorithm 4.3.5 to search for s.m.p.-candidates.

Every product, which is shorter than the s.m.p.-candidate and has averaged spectral radius greater or equal to $\tau \cdot \rho_c$, $\tau \in (0,1)$, is considered to be a nearly-s.m.p.. In our implementation we use a heuristic default value of $\tau = 0.9999$.

4.4.3 Approximate computation (4.4.3)

If we multiply the set of matrices $\tilde{\mathcal{A}}$ in step 4.4.1 (4.4.3) by a factor $0 < \delta < 1$, the modified invariant polytope algorithm 4.4.1 cannot return exact values for the JSR anymore; only up to a relative accuracy of δ. Indeed, if the modified invariant polytope algorithm 4.4.1 terminates, then $\|\tilde{A}_j\|_{\mathrm{co}_* V} \leq 1$ or, equivalently, $\|A_j\|_{\mathrm{co}_* V} \leq \delta^{-1} \cdot \rho_c$, $j \in \{1, \dots, J\}$, and thus $\mathrm{JSR}(\mathcal{A}) \leq \delta^{-1} \cdot \rho_c$. Nevertheless, there are cases where an approximate computation of the JSR is useful.

(*i*) If the dimension s of matrices is large, then the modified invariant polytope algorithm 4.4.1 will not terminate anyway in most cases, and thus only gives bounds for the JSR. An accuracy of $\delta \simeq 0.97$ speeds up the computation tremendously and the returned bounds are mostly better (at least in our numerical examples) than for $\delta = 1$.

(*ii*) If the s.m.p.s are not dominant, or there is an infinite number of dominant s.m.p.s, or the corresponding leading eigenvalues are not unique or simple, then the modified invariant polytope algorithm 4.4.1 will not terminate. In these cases, choosing $\delta \simeq 1 - 10^{-9}$ ensures that the invariant polytope algorithm 4.4.1 terminates and the obtained bounds are nearly the same as when $\delta = 1$.

(*iii*) If one is interested only whether $\mathrm{JSR}(\mathcal{A}) < C$ for some $C > 0$, one can choose $1 > \delta > \rho_c / C$ and the modified invariant polytope algorithm 4.4.1 terminates much faster.

4.4.4 Adding extra-vertices automatically (4.4.4)

Given some threshold $T > 0$, we compute the singular value decomposition of the cyclic root $\mathcal{H} = \left\{\alpha_1 v_1^{(0)}, \dots, \alpha_1 v_1^{(l(1)-1)}, \dots, \alpha_R v_R^{(1)}, \dots, \alpha_R v_R^{(l(R)-1)}\right\}$ and add all singular vectors $v_{R+1}, \dots, v_S$ corresponding to singular values which are in modulus less than $T > 0$ to the cyclic root $\mathcal{H}$. In view of Definition 4.2.23 these singular vectors are extra vertices. This strategy yields a good behaviour of the modified invariant polytope algorithm 4.4.1 in most of our examples. Note that with this choice of extra vertices the polytope $\mathrm{co}_* V_0$ always has non-empty interior.

In our implementation we use a heuristic default value of $T \simeq 0.1$.

4.4.5 Balancing of cyclic trees (4.4.5)

The original balancing procedure is described in [Guglielmi and Protasov, 2016, Section 3]. We present its improved version for the case of extra-vertices and nearly-s.m.p.s. If $\delta < 1$ no balancing is necessary as can be seen in the proof of Theorem 4.4.2. If $\delta = 1$, for v_r^*,

$r \in \{1, \dots, R\}$, i.e. the dual eigenvectors of the s.m.p.-candidates Π_r, and $q \in \{1, \dots, Q\}$ we define for $h \in \mathbb{N}$

$$p_{q,r} = \max\left\{ |(v_r^*, z)| : z \in \tilde{\mathcal{A}}^h \cdot \left\{ \left(\frac{\rho_c}{\rho_r}\right)^0 v_q^{(0)}, \dots, \left(\frac{\rho_c}{\rho_r}\right)^{l(q)-1} v_q^{(l(q)-1)} \right\} \right\}, \tag{4.4.12}$$

if v_q is an extra-vertex, i.e. $q \geq R+1$, then the maximum in (4.4.12) is only over $z \in \tilde{\mathcal{A}}^h \{v_q\}$. The factor $(\rho_c/\rho_r)^i$ ensures that all vertices of the cyclic root of nearly-s.m.p.s get the same weight in the computation. Note that, if v_q is the leading eigenvector of an s.m.p.-candidate, then $\rho_c/\rho_r = 1$. Now one has to determine numbers $\alpha_1, \dots, \alpha_Q > 0$ such that

$$\begin{cases} \alpha_q p_{q,r} & < & \alpha_r, & \text{whenever } v_q \text{ is the leading eigenvector of an s.m.p.-candidate and} \\ \alpha_q p_{q,r} & < & 1 & \text{otherwise.} \end{cases}.$$

Then, we multiply all vertices $v_q^{(i)}$, $q \in \{1, \dots, R\}$, $i \in \{0, \dots, l(q)-1\}$, and extra-vertices v_q, $q \in \{R+1, \dots, Q\}$, from the root $\mathcal{H}$ with the corresponding balancing factor α_q.

In our implementation we distinguish between extra-vertices and vertices from nearly-s.m.p.s., precisely we solve the following system with unknowns α_q, $q \in \{1, \dots, Q\}$ and $r \in \{1, \dots, R\}$,

$$\begin{cases} \alpha_q p_{q,r} & < & \alpha_r, & \text{whenever } v_q \text{ is the leading eigenvector of an s.m.p.-candidate,} \\ \alpha_q p_{q,r} & = & .999 \cdot \rho_q, & \text{whenever } v_q \text{ is the leading eigenvector of a nearly-s.m.p.,} \\ \alpha_q p_{q,r} & = & 1/100, & \text{whenever } v_q \text{ is an extra-vertex.} \end{cases}$$

The result [Guglielmi and Protasov, 2016, Theorem 3.3] ensures that the modified invariant polytope algorithm 4.4.1 terminates when started with both the s.m.p.-candidates, the nearly-s.m.p.s and the extra-vertices if and only if it terminates when started solely with the s.m.p.-candidates.

It was assumed (personal communication), at least for dimension $s = 1$, that the balancing factors for transition matrices occurring in subdivision theory are always equal to 1. It is not hard to find counterexamples in dimensions $s \geq 2$. The claim is not valid already in the univariate case, as the next example shows.

Example 4.4.4. Let S be the univariate subdivision scheme defined by the mask a and the dilation matrix M given by

$$a = \frac{1}{12}\begin{bmatrix} 3 & 3 & 4 & 3 & 3 & 4 & 3 & 3 & 4 & 3 & 3 \end{bmatrix}, \quad M = \begin{bmatrix} -3 \end{bmatrix}.$$

Taking the digit set $D = M[0,1] \cap \mathbb{Z} = \{-2, -1, 0\}$, we construct the set $\Omega_C = \{-4, -3, -2, -1, 0, 1\}$ and the corresponding transition matrices $T_{d,\Omega_C} = (a(\alpha - M\beta))_{\alpha,\beta \in \Omega_C}$, $d \in D$. The restriction of the transition matrices to the space $V_0(\Omega_C)$ of the first order differences with a basis

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

yields the set of matrices $\mathcal{T}|_{V_0(\Omega_C)} = \left\{T_{-2}|_{V_0(\Omega_C)}, T_{-1}|_{V_0(\Omega_C)}, T_0|_{V_0(\Omega_C)}\right\}$ given by

$$\frac{-1}{12}\begin{bmatrix} 0 & 0 & 0 & 3 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{-1}{12}\begin{bmatrix} 0 & 0 & 0 & 0 & 3 \\ 0 & 3 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{-1}{12}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix},$$

respectively. For the s.m.p.s $\Pi_1 = T_{-2}T_{-1}T_{-1}|_{V_0(\Omega_C)}$ and $\Pi_2 = T_{-1}T_{-1}T_0|_{V_0(\Omega_C)}$ with balancing factors α_1, $\alpha_2 = 9/10$ the modified invariant polytope algorithm 4.4.1 terminates after 4 iterations and the Hölder regularity of the basic limit function is $-\log_{|M|} \mathrm{JSR}(\mathcal{T}|_{V_0(\Omega_C)}) \simeq 0.9413$. With the balancing factors $\alpha_1 = \alpha_2 = 1$ the invariant polytope algorithm does not terminate. The basic limit function can be seen in Figure 4.6. △

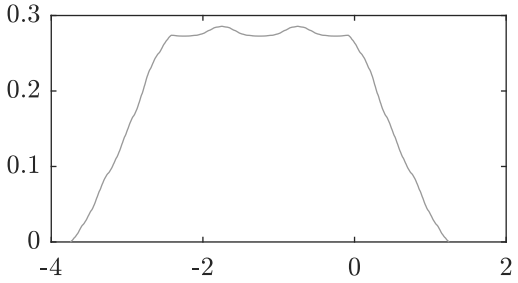


Figure 4.6: The basic limit function for the subdivision scheme from Example 4.4.4.

Example 4.4.5 shows the advantage of the new balancing procedure in connection with nearly-s.m.p.s.

Example 4.4.5. Given $A_1 = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$. The irreducible set $\mathcal{A} = \{A_1, A_2\}$ has A_2A_1 as an s.m.p. and $\rho(\mathcal{A}) \simeq 2.5396$. Assuming we start the modified invariant polytope algorithm 4.4.1 with this candidate and the nearly-s.m.p. A_2, with corresponding leading eigenvectors $v_1^{(0)} \simeq \begin{bmatrix} 0.9121 & 0.4100 \end{bmatrix}^T$, $v_2^{(0)} \simeq \begin{bmatrix} 0.4472 & 0.8944 \end{bmatrix}^T$ and leading dual eigenvectors $v_1^* \simeq \begin{bmatrix} 0.9958 & 0.2238 \end{bmatrix}^T$, $v_2^* \simeq \begin{bmatrix} 2.2361 & 0.0000 \end{bmatrix}^T$. For the balancing procedure as described in [Guglielmi and Protasov, 2016, Remark 3.7] we need to find numbers $\alpha_1, \alpha_2 > 0$ such that for some $h \in \mathbb{N}$, say $h = 10$, for

$$p_{1,2} = \max_{z \in \tilde{\mathcal{A}}^h \left\{ v_1^{(0)}, v_1^{(1)} \right\}} |(v_2^*, z)| = .0395\ldots,$$
$$p_{2,1} = \max_{z \in \tilde{\mathcal{A}}^h \left\{ v_2^{(0)} \right\}} |(v_1^*, z)| = .8196\ldots,$$

the following two inequalities hold:

$$\alpha_1 \cdot 2.0395\ldots = \alpha_1 p_{1,2} < \alpha_2$$
$$\alpha_2 \cdot 0.8196\ldots = \alpha_2 p_{2,1} < \alpha_1.$$

This is clearly impossible, since $2.0395\ldots \cdot 0.8196\ldots > 1$. By [Guglielmi and Protasov, 2016, Section 3], since there are no admissible balancing factors for $h = 10$, there are no admissible balancing factors for $h > 10$.

Nevertheless, since A_1A_2 is a dominant s.m.p., the modified invariant polytope algorithm 4.4.1 terminates if it is started only with A_1A_2. Thus, by [Guglielmi and Protasov, 2016, Theorem 4.1], there must exist balancing factors such that the algorithm terminates when started with $\Pi_1 = A_1A_2$ and $\Pi_2 = A_1$. Indeed, the algorithm terminates with $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$. △

4.4.6 Natural selection of vertices (4.4.6)

In the original invariant polytope algorithm 4.2.14, in every iteration all vertices generated in the foregoing iteration, which were not mapped inside the polytope, were used to construct new

vertices. In the modified invariant polytope algorithm 4.4.1 we only take a subset of those. We choose the vertices under the mild condition that

$$\text{for every } N \in \mathbb{N} \text{ there exists } K \in \mathbb{N},\ K \geq N \text{ such that } \bigcup_{n=0}^{N} \tilde{\mathcal{A}}^n V_0 \subseteq \operatorname{co}_* V_K, \tag{4.4.13}$$

where $\mathcal{A}^n$ are all matrix products of length n with matrices from $\mathcal{A}$. In other words, we do not forget any vertex to select. Two selection strategies turned out to work well:

(a) Choose those vertices which have the largest (for example the highest decile) norm $||V^+ \cdot ||_2$, where V^+ denotes the Moore-Penrose pseudo-inverse of V, and

(b) Choose those vertices whose parent vertex has largest norm with respect to the norm $|| \cdot ||_{\operatorname{co}_* V}$.

Strategy (a) reduces the number of vertices in V by roughly 20%, strategy (b) by roughly 10%. Since the intermediate bounds b_{k+1} for the JSR decrease very slowly when we use strategy (a) only, we iterate three times (a) and one time (b) in our implementation.

The natural selection of new vertices makes the modified invariant polytope algorithm 4.4.1 applicable for problems with a large number of matrices, since it ensures that the number of norms to be computed in each iteration is reasonably small. Of course, it does not substantially decrease the total number of norms we need to compute.

4.4.7 Simplified polytope (4.4.7)

In some examples the vertices constructed by the modified invariant polytope algorithm 4.4.1 are very near to each other, i.e. are at distances in the order of the machine epsilon. Those vertices are irrelevant for the size of the polytope; so we disregard them. This also protects against stability problems in the LP-programming part, since for simplices with vertices very near to each other LP-solvers perform very badly. This phenomenon happens frequently when there are multiple s.m.p.s. In our implementation we use a variable threshold in step 4.4.1 (4.4.7) when determining which vertices of the polytope we use in the computation of the norm.

It would be possible to choose a polytope $W_k(v)$, $v \in E_{k+1} \subseteq \tilde{\mathcal{A}} V_k \setminus V_k$, for each norm $||v||_{\operatorname{co}_* W_k(v)}$ we need to compute, since for each v we only need $s+1$ vectors from V_k to compute the norm $||v||_{\operatorname{co}_* V_k}$ exactly. Unfortunately we have no idea so far, how to select an adequate subset of V_k in a reasonable time, i.e. faster than the computation of the norm would take.

4.4.8 Parallelisation (4.4.8) & (4.4.9)

This is one of the main differences to the original implementation.

Instead of testing each vertex one after another, and adding it immediately to the set of vertices V_k if it is outside of the polytope, we compute the norms of all selected vertices from step 4.4.1 (4.4.6) with respect to the same polytope. Afterwards we add all vertices (which are outside of the polytope) at once to the set V_k.

This clearly leads to larger polytopes, in our examples the number of vertices increases by 10%. However, this is compensated by the fact that we can parallelise the computations of the norms. The speed-up is nearly linear in the number of available threads. Since the LP model does not change, we can speed up this part further by warm starting the LP problems, i.e. we reuse the solutions obtained from the computations of the other vertices. If there are no suitable candidates for the warm start, we still can speed up the LP problem by starting the search for the solution at the nearest vertex point of the polytope W. The speed-up from warm starting is roughly 50-70%.

4.4.9 Norm estimation (4.4.8)

Before computing the exact norm of a vector $v \in E_{k+1} \subseteq \tilde{\mathcal{A}}V_k \setminus V_k$, we try to determine its relative position (inside or outside of the polytope $\operatorname{co}_* V_k$) using the estimates in Lemma 4.4.6. If a vertex is proven to be inside or outside of the polytope, we do not have to compute its exact Minkowski norm anymore. If we cannot determine its position we compute the exact Minkowski norm. Unfortunately, the estimates in Lemma 4.4.6 are quite rough and fail to determine the position for most vertices, except in the positive case (P) where Lemma 4.4.6 (v) gives very accurate estimates.

Lemma 4.4.6. *Given $V \subseteq \mathbb{R}^s$ and $x \in \mathbb{R}^s$.*

(i) *If W are the vertices of another polytope with non-empty interior such that $\operatorname{co}_* W \subseteq \operatorname{co}_* V$, then $||\cdot||_{\operatorname{co}_* V} \leq ||\cdot||_{\operatorname{co}_* W}$.*

(ii) *$||x||_{\operatorname{co}_s V} \leq ||t||_1$ for all $t \in \mathbb{R}^{\#V}$ such that $Vt = x$.*

(iii) *$||x||_{\operatorname{co}_s V} \geq ||V^+ x||_2$, where V^+ is the Moore-Penrose pseudo-inverse of V.*

(iv) *If there exists $w \in \mathbb{R}^s$ such that $|(w, v)| < |(w, x)|$ for all $v \in V \cup -V$, then $x \notin \operatorname{co}_s V$.*

(v) *If $V \subseteq \mathbb{R}^s_+$, $x \in \mathbb{R}^s_+$, and there exists $v \in V$ such that $x_l \leq v_l$ for all $l \in \{1, \ldots, s\}$, then $x \in \operatorname{co}_- V$.*

Proof. (i) This immediately follows from Definition 4.2.6 of the Minkowski norm.

(ii) Let $x \in \operatorname{co}_s V$ and $t \in \mathbb{R}^{\#V}$ such that $x = Vt$. Define $\tilde{x} = \frac{x}{||x||_{\operatorname{co}_s V}} \in \partial \operatorname{co}_s V$ and $\tilde{t} = \frac{t}{||x||_{\operatorname{co}_s V}}$. It follows that $\tilde{x} = V\tilde{t}$ with $\left|\left|\tilde{t}\right|\right|_1 \geq 1$. Indeed, $\left|\left|\tilde{t}\right|\right|_1 < 1$ would imply that $\tilde{x} \in (\operatorname{co}_s V)^\circ$. Clearly, $1 = ||\tilde{x}||_{\operatorname{co}_s V} \leq \left|\left|\tilde{t}\right|\right|_1$ and $||x||_{\operatorname{co}_s V} \leq ||t||_1$.

(iii) Let $x \in \operatorname{co}_s V$ and $t \in \mathbb{R}^{\#V}$ such that $x = Vt$. It follows that $||t||_1 \geq ||t||_2 \geq ||V^+ x||_2$ because, by construction of the Moore-Penrose pseudo inverse, $V^+ x$ is the unique solution to $Vt = x$ with minimum 2-norm. Finally, by the proof of (ii), $||x||_{\operatorname{co}_s V} = \min_{t \in \mathbb{R}^{\#V} : Vt = x} ||t||_1 \geq \left|\left|V^+ x\right|\right|_2$.

(iv) If $|(w, v)| < |(w, x)|$ for all $v \in V$, then there exists a hyperplane which separates the point x and the polytope V. From this the claim follows.

(v) Let $x \in \mathbb{R}^s_+$ with $x_l \leq v_l$, $l \in \{1, \ldots, s\}$, for some $v \in V$. Therefore, $z = v - x \in \mathbb{R}^s_+$. Thus, $x = v - z$ with $v \in V \subseteq \operatorname{co}_- V$, $z \in \mathbb{R}^s_+$, by (4.2.5), belongs to $x \in \operatorname{co}_- V$. □

Remark 4.4.7. Estimate 4.4.6 (v) uses the fact that the norms $||\cdot||_{\operatorname{co}_- V}$ are orthant monotonic. It would be interesting to know whether and when Minkowski norms $||\cdot||_{\operatorname{co}_s V}$ are orthant monotonic. This would allow to transfer the estimate 4.4.6 (v) to the case (R).

4.4.10 Intermediate bounds for the joint spectral radius (4.4.10)

The original method to compute intermediate upper bounds for the JSR is described in [Guglielmi and Protasov, 2013, Section 2.1]. This method sometimes may return wrong intermediate bounds. We present an improved method to compute these bounds. Thus, the modified invariant polytope algorithm 4.4.1 can be used to give estimates for the JSR of finite sets of matrices.

Definition 4.4.8. We define $b_0 = \infty$ and b_{k+1}, $k \in \mathbb{N}_0$ by

$$b_{k+1} = \begin{cases} b_k, & \text{if} \quad E_{k+1} \subsetneqq \tilde{\mathcal{A}}V_k \setminus V_k \quad \text{and} \\ \min\left\{b_k, \max\left\{1, N(v)(1-\epsilon)^{-1} : v \in E_{k+1}\right\}\right\}, & \text{if} \quad E_{k+1} = \tilde{\mathcal{A}}V_k \setminus V_k. \end{cases}$$

Lemma 4.4.9. *For all $k \in \mathbb{N}_0$, $\mathrm{JSR}(\mathcal{A}) \in [\rho_c,\ \delta^{-1} \cdot b_{k+1} \cdot \rho_c]$.*

Proof. The lower bound follows by Theorem 4.1.2.

For the upper bound, by 4.4.1 (4.4.3), it suffices to show that $\mathrm{JSR}(\tilde{\mathcal{A}}) \leq b_{k+1}$ for all $k \in \mathbb{N}_0$. The claim is true for b_0 since $\mathrm{JSR}(\tilde{\mathcal{A}}) < \infty$. Now assume the claim is true for $k-1$, $k \in \mathbb{N}$. We show that it is true for k. If $E_{k+1} \subsetneqq \tilde{\mathcal{A}}V_k \setminus V_k$, then $b_{k+1} = b_k$ and the claim follows for this case.

Hence, we assume that $E_{k+1} = \mathcal{A}V_k \setminus V_k$. If the algorithm terminates in this iteration, then $\mathrm{JSR}(\tilde{\mathcal{A}}) \leq 1$. Thus, the claim follows for this case, since $b_{k+1} \geq 1$, $k \in \mathbb{N}_0$, by definition of b_{k+1}.

Hence, we assume that the algorithm does not terminate in step k. which implies that there exists $v \in E_{k+1}$ such that $N(v)(1-\epsilon)^{-1} > 1$. Consequently, with $\tilde{b}_{k+1} = \max\{N(v)(1-\epsilon)^{-1} : v \in E_{k+1}\}$,

$$\tilde{b}_{k+1} = \max\{1,\ N(v)(1-\epsilon)^{-1} : v \in E_{k+1}\}.$$

If $\tilde{b}_{k+1} \geq b_k$, then $\mathrm{JSR}(\tilde{\mathcal{A}}) \leq b_k \leq \tilde{b}_{k+1}$ and the claim follows for this case.

Hence, we assume that $\tilde{b}_{k+1} < b_k$. We have to show that $||v||_{\mathrm{co}_* V_k} \leq \tilde{b}_{k+1}$ for all $v \in \tilde{\mathcal{A}}V_k$. If $v \in V_k$, then $||v||_{\mathrm{co}_* V_k} \leq 1 \leq b_{k+1}$, because $V_k \subseteq \mathrm{co}_* V_k$ and $b_{k+1} \geq 1$. If $v \in \tilde{\mathcal{A}}V_k \setminus V_k = E_{k+1}$ and the value of $N(v)$ was computed in iteration $\tilde{k} \leq k$, $\tilde{k} \in \mathbb{N}_0$, then $||v||_{\mathrm{co}_* V_k} \leq ||v||_{\mathrm{co}_* V_{\tilde{k}}} \leq ||v||_{\mathrm{co}_* W_{\tilde{k}}} = N(v)(1-\epsilon)^{-1} \leq b_{k+1}$, where $W_{\tilde{k}}$ is defined in 4.4.1 (4.4.7).

Therefore, $\mathrm{JSR}(\tilde{\mathcal{A}}) \leq b_{k+1}$. □

We now discuss the problem which can arise when intermediate bounds are computed. As already mentioned, in the original invariant polytope algorithm 4.2.14, a vertex is immediately added to the set of vertices V of the polytope $\mathrm{co}_* V$, when this vertex lies outside of the polytope. Assume that the set $\tilde{\mathcal{A}}$ consists of two matrices $\tilde{A}_1, \tilde{A}_2 \in \mathbb{R}^{s \times s}$ and we have to test whether all children of $v \in V$ are inside or outside the polytope $\mathrm{co}_* V$. Assume now, $\|\tilde{A}_1 v\|_{\mathrm{co}_* V} = N_1 > 1$, and thus $\tilde{A}_1 v$ is added to the set of vertices V, and assume that $\|\tilde{A}_2 v\|_{\mathrm{co}_* V \cup \tilde{A}_1 v} = N_2 > 1$. Let $N = \max\{N_1, N_2\}$. Now, V is not an invariant polytope for the matrices $N^{-1}\tilde{\mathcal{A}}$ in general, and thus $\mathrm{JSR}(\tilde{\mathcal{A}}) \not\leq N$ in general. Indeed, although we have that $\|\tilde{A}_1 v\|_{\mathrm{co}_* V} = N_1 \leq N$, for the other vertex $\tilde{A}_2 v$ we only know that $\left|\left|\tilde{A}_2 v\right|\right|_{\mathrm{co}_* V} \geq \left|\left|\tilde{A}_2 v\right|\right|_{\mathrm{co}_* V \cup \tilde{A}_1 v} = N_2 \leq N$.

4.4.11 New stopping criterion (4.4.11)

If the candidates Π_r, $r \in \{1, \ldots, R\}$, are s.m.p.s, then all intermediately occurring matrix products in the modified invariant polytope algorithm 4.4.1 have spectral radius less than 1. Otherwise, if the candidates are not s.m.p.s, then it still can happen that all intermediately occurring matrix products have spectral radius less than 1 and the modified invariant polytope algorithm 4.4.1 never stops, see Example 4.4.12. However, this never happened in any non-artificial example.

Furthermore, products with larger averaged spectral radius always occurred very fast. Thus, from a practical point of view, (4.4.1) (4.4.11) is a better way to check whether the candidates are s.m.p.s or not, than the method described in [Guglielmi and Protasov, 2013, Proposition 2]. The method of Guglielmi and Protasov [2013, Proposition 2] on the other hand, is fail-proof and eventually always strikes whenever an s.m.p.-candidate is not an s.m.p.. In our implementation `tjsr` of the modified invariant polytope algorithm [Mejstrik, 2018b] we, thus, implement both stopping criteria. To compute the spectral radii, we use the Matlab function `eig`. This may not

be the fastest available procedure, but it is fast enough compared to the time the computation of the norms take.

We illustrate how the stopping criterion 4.4.1 (4.4.11) may fail.

Definition 4.4.10. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1, \dots, J\}$ and $\eta \geq 0$. We define the set $\mathcal{M}_\eta$ by

$$\mathcal{M}_\eta = \left\{(j_n)_n \in \{1, \dots, J\}^{\mathbb{N}} : \rho(\tilde{A}_{j_m} \cdots \tilde{A}_{j_1})^{1/m} \leq \eta \text{ for all } m \in \mathbb{N}\right\}.$$

For $\eta = 1$, the products $\tilde{A}_{j_n} \cdots \tilde{A}_{j_1}$, $(j_n)_n \in \mathcal{M}_1$, $\tilde{A}_j \in \tilde{\mathcal{A}}$, are exactly the products occurring in the modified invariant polytope algorithm 4.4.1 until the stopping criterion 4.4.1 (4.4.11) strikes. The hope would be that the norms of the products in this sequence stay bounded, i.e. there exists $C > 0$ such that $\left|\left|\tilde{A}_{j_n} \cdots \tilde{A}_{j_1}\right|\right| < C$ for all $n \in \mathbb{N}$. Unfortunately, this is wrong. Even for $\eta = 0$, as in Example 4.4.11, the norms of the products may go to infinity

Example 4.4.11. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $C = A^T$. Clearly, $\mathrm{JSR}(\{A, B, C\}) = 2$. Indeed, $\rho(A) = 2$ and $||A||_2 = ||B||_2 = ||C||_2 = 2$. The matrices of the sequence $(A^n B)_{n\in\mathbb{N}}$ have spectral radius zero, yet the sequence $||A^n B||$ goes to infinity as $n \to \infty$.

Note that, although the set of matrices $\{A, B, C\}$ is irreducible, the sequence $(A^n B)_{n\in\mathbb{N}}$ only consists of products of matrices of the reducible set $\{A, B\}$. △

Unfortunately, the spectral radius of the (wrong) s.m.p.-candidate B in Example 4.4.11 is zero. Thus, by Remark 4.2.21, the invariant polytope algorithm cannot be used. Therefore, we need another example to prove that the stopping criterion may fail.

Example 4.4.12. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \frac{3}{4}A$. Then $\mathrm{JSR}(\{A, B\}) = 1$, $\rho(A^n B) < 1$ for all $n \in \mathbb{N}$, but $||A^n B|| \to \infty$ as $n \to \infty$. Indeed, for $n \in \mathbb{N}$, $A^n B = \frac{3}{4}\begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$, and thus it follows that $\rho(A^n B) = \frac{3}{4}$ and $||A^n B||_1 = \frac{6+3n}{4}$.

Now define the irreducible set $\mathcal{A} = \left\{A, A^T, B, B^T\right\}$. Then, the modified invariant polytope algorithm 4.4.1, together with an (unlucky) version of the natural selection of vertices procedure and started with the (wrong) s.m.p.-candidates $\Pi_1 = A$, $\Pi_2 = A^T$, and root vectors $v_1^{(0)} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $v_2^{(0)} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, can construct an arbitrarily large polytope, solely with products whose spectral radius is strictly less than one. Indeed, for $n \in \mathbb{N}$, applying the sequence of products $A^n B$ to the starting vector v_2 we get the sequence of vectors $A^n B v_2 = \begin{bmatrix} 3n/4 & 3/4 \end{bmatrix}^T$, $n \in \mathbb{N}$. The same calculation shows that $(A^T)^n B^T v_1 = \begin{bmatrix} 3/4 & 3n/4 \end{bmatrix}^T$, $n \in \mathbb{N}$. These two sequences generate an unbounded polytope. △

Proof for Theorem 4.4.2

Proof. First note the following. If $V_N^{orig}, V_N^{mod} \subseteq \mathbb{R}^s$, $N \in \mathbb{N}$, denote the vertices constructed in the N^{th} iteration of the original invariant polytope algorithm 4.2.14 and the modified invariant polytope algorithm 4.4.1, respectively, then, due to the construction of the original invariant polytope algorithm 4.2.14,

$$\mathrm{co}_*\, V_N^{orig} = \bigcup_{n=0}^{N} \tilde{\mathcal{A}}^n V_0 \tag{4.4.14}$$

and, due to the construction of the modified invariant polytope algorithm 4.4.1,

$$\mathrm{co}_*\, V_N^{mod} \subseteq \bigcup_{n=0}^{N} \tilde{\mathcal{A}}^n V_0. \tag{4.4.15}$$

(*i*) Let $\delta = 1$. First assume that the original invariant polytope algorithm 4.2.14 terminates at depth $N \in \mathbb{N}$ with vertices V_N^{orig}. Thus, by the invariance of $\operatorname{co}_* V_N^{orig}$, we have $\tilde{\mathcal{A}} \operatorname{co}_* V_N^{orig} \subseteq \operatorname{co}_* V_N^{orig}$. By (4.4.14) and (4.4.13), there exists $K \in \mathbb{N}$, $K \geq N$, such that

$$\operatorname{co}_* V_N^{orig} = \operatorname{co}_* \bigcup_{n=0}^{N} \tilde{\mathcal{A}}^n V_0 \subseteq \operatorname{co}_* V_K^{mod},$$

where V_K^{mod} is the set of vertices constructed in the K^{th} iteration of the modified invariant polytope algorithm 4.4.1. We claim that $\operatorname{co}_* V_K^{mod}$ is an invariant polytope. By (4.4.15),

$$\operatorname{co}_* V_K^{mod} \subseteq \operatorname{co}_* \bigcup_{k=0}^{K} \tilde{\mathcal{A}}^k V_0.$$

By the invariance property of the polytope $\operatorname{co}_* V_N^{orig}$ and by $K \geq N$,

$$\operatorname{co}_* \bigcup_{k=0}^{K} \tilde{\mathcal{A}}^k V_0 = \operatorname{co}_* V_N^{orig}.$$

It follows that $\operatorname{co}_* V_N^{orig} = \operatorname{co}_* V_K^{mod}$, and thus $\operatorname{co}_* V_K^{mod}$ is an invariant polytope.

The other direction follows similarly. Let $\delta = 1$. Assume that the modified invariant polytope algorithm 4.4.1 terminates at depth $N \in \mathbb{N}$ with vertices V_N^{mod}. Thus, by the invariance of $\operatorname{co}_* V_N^{mod}$, we have $\tilde{\mathcal{A}} \operatorname{co}_* V_N^{mod} \subseteq \operatorname{co}_* V_N^{mod}$. We denote by V_N^{orig} the vertices constructed in the N^{th} iteration of the original invariant polytope algorithm 4.2.14. By (4.4.14),

$$\operatorname{co}_* V_N^{orig} = \operatorname{co}_* \bigcup_{n=0}^{N} \tilde{\mathcal{A}}^n V_0.$$

We claim that $\operatorname{co}_* V_N^{orig}$ is an invariant polytope. By the invariance property of the polytope $\operatorname{co}_* V_N^{mod}$,

$$\operatorname{co}_* V_N^{mod} \subseteq \operatorname{co}_* \bigcup_{n=0}^{N} \tilde{\mathcal{A}}^n V_0 \subseteq \operatorname{co}_* \bigcup_{n=0}^{N} \tilde{\mathcal{A}}^n V_N = \operatorname{co}_* V_N^{mod}.$$

It follows that $\operatorname{co}_* V_N^{orig} = \operatorname{co}_* V_N^{mod}$, and, therefore, $\operatorname{co}_* V_N^{orig}$ is an invariant polytope.

(*ii*) Let $0 < \delta < 1$ and denote with ρ_c is the averaged spectral radius of the s.m.p.-candidates. Assume that $\operatorname{JSR}(\mathcal{A}) \leq \delta^{-1} \cdot \rho_c$ or, equivalently by (4.4.3), $\operatorname{JSR}(\tilde{\mathcal{A}}) < 1$.

By Lemma 4.1.1, there exists a norm $||\cdot||$ such that $\left|\left|\tilde{A}_j\right|\right| < 1$ for all $j \in \{1, \ldots, J\}$. Thus, applying the matrices $\tilde{A}_j$ several times to vectors $v \in V_0$, we eventually map them to $(\operatorname{co}_* V_0)^\circ$. Note that the interior of $\operatorname{co}_* V_0$ is non-empty due to Section 4.4.4. Hence, the modified invariant polytope algorithm 4.4.1 eventually constructs an invariant polytope and terminates.

(*iii*) This we have shown in Lemma 4.4.9. □

4.5 Applications and numerical results

In this section we illustrate the modified Gripenberg algorithm 4.3.5 and the modified invariant polytope algorithm 4.4.1 with numerical examples. For our tests we use matrices from standard applications, as well as random matrices. Furthermore, we try to repeat tests which were performed in similar papers [Moision, Orlitsky and Siegel, 2001; Blondel, Jungers and Protasov, 2006; Blondel and Chang, 2011, 2013; Guglielmi and Protasov, 2013, 2016].

The parameters for the fast algorithms are chosen such that they terminate after a reasonably short time; i.e. less then few minutes. For the modified invariant polytope algorithm 4.4.1 the input parameters are chosen such that the algorithm terminates at all, hopefully in shortest time. We do not report the exact input parameters, since we believe they are of no value for the reader. The tests are performed using an Intel Core i5-4670S@3.8GHz, 8GB RAM with the software Matlab R2017a, t-packages v1.0 [Mejstrik, 2018b], JSR-Toolbox v1.2b [Hendrickx, Jungers and Vankeerberghen, 2011], SeDuMi Toolbox v1.32 [Sturm, 1999] and Gurobi solver v8.0 [Gurobi Optimization, 2018].

Remark 4.5.1. All matrix test sets are constructed with the function `tgallery` [Mejstrik, 2018b].

Modified invariant polytope algorithm 4.4.1

To summarize, we can say that the single-threaded modified invariant polytope algorithm 4.4.1 is roughly three times faster than the original invariant polytope algorithm. If the dimension of the matrices is sufficiently large, the modified invariant polytope algorithm 4.4.1 scales nearly linearly with the number of available threads. Thus, on a standard PC with 8 cores, the modified invariant polytope algorithm 4.4.1 is roughly 25 times faster. More precisely

- for pairs of random matrices the modified invariant polytope algorithm 4.4.1 reports the exact value of the JSR in reasonable time up to dimension 25, compared to dimension 20 for the original invariant polytope algorithm 4.4.1,
- for pairs of random matrices with non-negative entries the modified invariant polytope algorithm 4.4.1 reports the exact value of the JSR in reasonable time up to dimension 3000, compared to dimension 1000 for the original invariant polytope algorithm 4.4.1,
- for Daubechies matrices the modified invariant polytope algorithm reports the exact value of the JSR in reasonable time up to dimension 42, compared to dimension 20 for the original invariant polytope algorithm 4.4.1, and
- for the binary matrices arising in the context of the computation of the capacity of codes avoiding certain forbidden difference patterns, the algorithm works well only up to dimension 16.

Modified Gripenberg algorithm 4.3.5

The modified Gripenberg algorithm 4.3.5 finds in almost all cases an s.m.p.. Thus, for fast estimates of the JSR, the modified Gripenberg algorithm 4.3.5 may be used independently.

Clearly, since the computation of the JSR is NP-hard, there are sets of matrices for which algorithm 4.3.5 fails and we report mostly these cases together with a comparison with other known algorithms: These are the random modified Gripenberg algorithm 4.3.5 described in Remark 4.3.3, the Gripenberg algorithm itself, described in Section 4.3.1, the modified invariant polytope algorithm 4.4.1, described in Section 4.3.2, and the genetic algorithm by Blondel and Chang [2011], implemented by Chang [2018], which is a Monte Carlo algorithm.

At least in our test runs, the modified Gripenberg algorithm 4.3.5 performs best, in the sense that in most cases it returns a correct s.m.p. faster than all other algorithms. More precisely,

- for long s.m.p.s the modified Gripenberg algorithm 4.3.5 performs best,
- for large sets of matrices the genetic algorithm and the modified invariant polytope algorithm 4.4.1 perform best on average.

4.5.1 Randomly generated matrices

Matrices with normally distributed values We first illustrate the behaviour of the modified invariant polytope algorithm 4.4.1 for pairs of matrices of dimensions 2 to 20 with normally distributed values whose

(Test A) matrices have the same 2-norm,

(Test B) matrices have the same spectral radius, and

(Test C) matrices have the same spectral radius and $\delta = 0.99$,

where δ is the parameter controlling the accuracy of the algorithm, see Section 4.4.3 (4.4.3). We print the median values for these three examples, since there are always some outliers in tests with $\delta = 1$. *The average values are roughly 100 times larger.* We see in Figure 4.7 that the modified invariant polytope algorithm 4.4.1 is applicable for pairs of random matrices up to dimension 25, for which it takes roughly one weekend to terminate. For $\delta = 0.95$ the modified invariant polytope algorithm 4.4.1 is comparable to Gripenbergs algorithm and we do not print the exact test results.

The test with the original invariant polytope algorithm with matrices of equal 2-norm shows that the modified invariant polytope algorithm 4.4.1 produces polytopes with roughly twice as many vertices; yet it still works for matrices of dimension 20, see [Guglielmi and Protasov, 2013, Figure 2] .

$J = 2$, $\#test = 20$, *median values*						
	(Test A) $\delta = 1$ *equal 2-norm*		*(Test B)* $\delta = 1$ *equal spectral radius*		*(Test C)* $\delta = 0.99$ *equal spectral radius*	
dim	*time*	$\#V$	*time*	$\#V$	*time*	$\#V$
2	1.1 *s*	5·2	1.2 *s*	6·2	0.2 *s*	5·2
4	1.4 *s*	17·2	1.8 *s*	77·2	0.8 *s*	19·2
6	2.0 *s*	47·2	2.5 *s*	130·2	1.5 *s*	47·2
8	2.5 *s*	100·2	3.9 *s*	220·2	2.1 *s*	98·2
10	4.9 *s*	270·2	5.1 *s*	320·2	3.3 *s*	220·2
12	4.7 *s*	280·2	11 *s*	770·2	6.6 *s*	570·2
14	8.4 *s*	510·2	21 *s*	1100·2	12 *s*	800·2
16	25 *s*	1100·2	33 *s*	1400·2	25 *s*	1000·2
18	90 *s*	2100·2	200 *s*	2500·2	44 *s*	1600·2
20	295 *s*	3100·2	5000 *s*	6200·2	800 *s*	3900·2

Figure 4.7: Computation of the JSR with the modified invariant polytope algorithm 4.4.1 for pairs of random matrices with equal 2-norm or equal spectral radius.
δ: accuracy parameter for the modified invariant polytope algorithm 4.4.1, *dim*: dimension of the matrices, $\#V$: number of vertices of the invariant polytope, *time*: time needed to compute the invariant polytope, J: number of matrices, $\#test$: number of test runs.

Matrices with non-negative values Random matrices with non-negative entries are a worthy test case, since the computation of the invariant polytope (i.e. the main loop in the modified invariant polytope algorithm 4.4.1) always finishes after a few seconds, nearly regardless of the dimension. Since the implementation `tjsr` [Mejstrik, 2018b] is not optimized for such high dimensions, our implementation of the modified invariant polytope algorithm 4.4.1 still needs some minutes to terminate, mostly due to the preprocessing steps (4.4.2) – (4.4.5).

For sparse matrices with non-negative entries, the modified invariant polytope algorithm 4.4.1 performs slightly worse; yet it is still applicable up to dimension 3000 or higher. Again, it is very likely that the modified invariant polytope algorithm 4.4.1 still works for even larger matrices if the implementation `tjsr` were optimized for such matrices, see Figure 4.8 for the results. We again give the median values. The average values for these cases are roughly 10% higher.

A better benchmark with non-negative matrices is presented in Section 4.5.3.

$J = 2$, $\#test = 20$, *non-negative entries, equal spectral radius, median values*								
	0% sparsity		*90% sparsity*		*98% sparsity*		*99% sparsity*	
$dim^{\dagger}$	*time*	$\#V^{\dagger\dagger}$	*time*	$\#V^{\dagger\dagger}$	*time*	$\#V^{\dagger\dagger\dagger}$	*time*	$\#V^{\dagger\dagger\dagger}$
20	0.3 *s*	7	1.7 *s*	42				
50	0.3 *s*	8	1.6 *s*	50	2.2 *s*	50		
100	0.4 *s*	8	0.8 *s*	25	17 *s*	1300		
200	0.5 *s*	8	1.0 *s*	23	5.0 *s*	220	110 *s*	2600
500	1.2 *s*	8	1.8 *s*	16	7.7 *s*	90	26 *s*	310
1000	6.3 *s*	8	11 *s*	16	30 *s*	45	72 *s*	110
2000	35 *s*	8	72 *s*	16	35 *s*	8	290 *s*	64

Figure 4.8: Computation of the JSR with the modified invariant polytope algorithm 4.4.1 for random pairs of matrices with non-negative entries and equal spectral radius.
dim: dimension of the matrices, J: number of matrices, $\#test$: number of test runs.
time: time needed to compute the invariant polytope, $\#V$: number of vertices of the invariant polytope.
†Since the matrices are random, most of the sparse matrices have non-trivial invariant subspaces which reduces the effective dimension of the matrices by roughly 10%.
††Most cones have 8 or 16 vertices, because the algorithm terminates after 3 or 4 iterations. The algorithm does not check whether all of these vertices are really in the boundary of the polytope. ††† Note that the number of vertices decreases with the dimension. We have no explanation for this phenomenon yet.

Matrices with equally distributed values In Figure 4.9 we see how the modified Gripenberg algorithm 4.3.5 performs on random matrices. These matrices have equally distributed values in $[-5,\ 5]$ to mimic the test in [Blondel and Chang, 2011, Section 4.2]. Interestingly, the genetic algorithm performs very badly, as does the *random* modified Gripenberg algorithm. The modified Gripenberg algorithm 4.3.5 performs in the best possible way, i.e. it always finds an s.m.p.. We report the *success*-rate in percent, i.e. how often the algorithms did find an s.m.p..

Remark 4.5.2. The random matrices for the tests in Figure 4.7 are generated with the function `tgallery('rand_gauss',dim,J,'norm')` and `tgallery('rand_gauss',dim,J,'rho')`. The command `tgallery('rand_equal',dim,J,'pos','sparse',sparsity)` generates the random matrices for the tests in Figure 4.8.

$\#tests = 100$, *equal spectral radius*						
	$J = 2$, $dim = 2$		$J = 4$, $dim = 4$		$J = 8$, $dim = 8$	
Algorithm	*success*	*time*	*success*	*time*	*success*	*time*
mod. invariant polytope	100%	1.1 s	100%	4.3 s	100%	40.0 s
mod. Gripenberg	100%	1.9 s	100%	4.1 s	100%	5.4 s
random Gripenberg	100%	1.8 s	99%	3.8 s	82%	4.3 s
Gripenberg	100%	3.8 s	100%	20.3 s	100%	82.1 s
brute force	100%	180 s	98%	180.0 s	74%	180.0 s
genetic	100%	7.1 s	97%	9.3 s	87%	12.0 s

Figure 4.9: Performance of the modified Gripenberg algorithm 4.3.5 for sets of random matrices with equally distributed values and equal spectral radius.
dim: dimension of the matrices, J: number of matrices, *success*: percentage of how often a correct s.m.p. is found. $\#test$: number of test runs. *time*: time needed by the algorithm.

4.5.2 Handpicked generic matrices

Example 4.5.3. Let

$$X_1 = \begin{bmatrix} \frac{15}{92} & \frac{-73}{79} \\ \frac{56}{59} & \frac{89}{118} \end{bmatrix}, \; X_2 = \begin{bmatrix} \frac{-231}{241} & \frac{-143}{219} \\ \frac{103}{153} & \frac{-38}{65} \end{bmatrix}.$$

The set $\mathcal{X} = \{X_1, X_2\}$ has an s.m.p. of length 119 with averaged spectral radius $\mathrm{JSR}(\mathcal{X}) \simeq 1.01179$. The Gripenberg algorithm finds an s.m.p. after an evaluation of ∼630.000 products, taking roughly ten minutes. Both the modified Gripenberg algorithm 4.3.5, as well as the genetic algorithm fail. The modified invariant polytope algorithm 4.4.1 finds an s.m.p. in less than one minute. The test results are in Figure 4.10 (left side). The invariant polytope constructed by the modified invariant polytope algorithm 4.4.1 can be seen in Figure 4.10 (right side). △

Remark 4.5.4. The command `tgallery('mejstrik_119')` generates the set $\mathcal{X}$.

Test set	*Algorithm*	*Lower bd.*	*time*
$\mathcal{X}$	mod. invariant polytope	1.01179 . . .	40 s
$J = 2$	mod. Gripenberg	1.01130 . . .	4 s
$dim = 2$	random Gripenberg	1.01172 . . .	10 s
	Gripenberg	1.01179 . . .	580 s
	genetic	1.01130 . . .	8 s

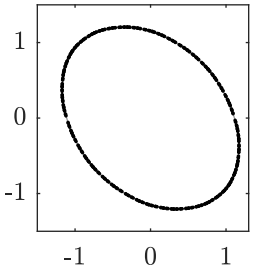


Figure 4.10: Left side: For the test set $\mathcal{X}$ from Example 4.5.3 all fast algorithms fail, since they do not find a correct s.m.p..
dim: dimension of the matrices, *lower bd.*: computed lower bound for the JSR, J: number of matrices, *time*: time needed by the algorithm.
Right side: Invariant polytope constructed for the set $\mathcal{X}$.

Example 4.5.5 is of interest because it is a rather simple family of two matrices with arbitrary long s.m.p.s.

Example 4.5.5. Let $n \in \mathbb{N}$, $C_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $C_n = \begin{bmatrix} 0 & 0 \\ \frac{1}{n}e^{1+\frac{1}{n}} & 0 \end{bmatrix}$. We show in Section 4.6 that $\mathrm{JSR}(\mathcal{C}_n) = \rho(C_0^n C_n)^{1/(n+1)} = e^{1/n}$ for $\mathcal{C}_n = \{C_0, C_n\}$.

The genetic algorithm fails for most matrices of this family. All other algorithms report the correct s.m.p. in less than 5 *s*. The test results are in Figure 4.11. △

Test set	*Algorithm*	*Lower bd.*	*time*
$\mathcal{C}_{15}$	mod. invariant polytope	1.0689...	1.7 *s*
$J = 2$	mod. Gripenberg	1.0689...	3.3 *s*
$dim = 2$	random Gripenberg	1.0689...	3.2 *s*
$s.m.p. = C_0^{15}C_{15}$	Gripenberg	1.0689...	0.1 *s*
	genetic	1.0689...	7.0 *s*
$\mathcal{C}_{30}$	mod. invariant polytope	1.0338...	2.5 *s*
$J = 2$	mod. Gripenberg	1.0338...	4.0 *s*
$dim = 2$	random Gripenberg	1.0338...	4.3 *s*
$s.m.p. = C_0^{30}C_{30}$	Gripenberg	1.0338...	0.1 *s*
	genetic	1.0215...	6.6 *s*
$\mathcal{C}_{60}$	mod. invariant polytope	1.0168...	4.0 *s*
$J = 2$	mod. Gripenberg	1.0168...	3.1 *s*
$dim = 2$	random Gripenberg	1.0168...	4.3 *s*
$s.m.p. = C_0^{60}C_{60}$	Gripenberg	1.0168...	0.1 *s*
	genetic	1.0000...	6.3 *s*

Figure 4.11: For the test sets $\mathcal{C}_n$ from Example 4.5.5 the genetic algorithm mostly fails.
dim: dimension of the matrices, *lower bd.*: computed lower bound for the JSR, J: number of matrices, *s.m.p.*: an s.m.p., *time*: time needed by the algorithm

Remark 4.5.6. The command `tgallery('mejstrik_Cn',n)` generates the sets $\mathcal{C}_n$ from Example 4.5.5.

4.5.3 Capacity of codes with forbidden difference sets

In some electromagnetic recording systems, the bit error rate is often dominated by a small set of certain *forbidden difference words* D. Thus, one needs to construct sets of allowed words all of whose possible differences do not yield such a forbidden word. Clearly, one wants codes with few constrains on the possible words. We are interested in how constraining a given set of forbidden difference words D is, which we denote as the *capacity* $\operatorname{cap} D \in [0, 1]$ which we define next. The larger the capacity, the better.

Definition 4.5.7 ([Moision, Orlitsky and Siegel, 2001, Section 3], [Blondel, Jungers and Protasov, 2006, Section V])**.** Let $K \in \mathbb{N}$.

(*i*) We define a *word (of length* $n \in \mathbb{N}$*)* as a vector in $\{0, \dots, K\}^n$. The *concatenation* of two words $u \in \{0, \dots, K\}^{n_u}$, $v \in \{0, \dots, K\}^{n_v}$, $n_u, n_v \in \mathbb{N}$, is denoted by $uv \in \{0, \dots, K\}^{n_u+n_v}$.

(*ii*) For two words $u, v \in \{1, \dots, K\}^n$, we define the *difference* by $u - v \in \{-K, \dots, K\}^n$. A set of *forbidden differences* is $D \subseteq \{-K, \dots, K\}^n$, $n \in \mathbb{N}$, is a finite set of difference words.

(*iii*) By $\delta_n(D)$, $n \in \mathbb{N}$, $D \subseteq \{-K, \dots, K\}^m$, $m \in \mathbb{N}$, we denote the largest cardinality of sets of words of length n whose differences avoid the forbidden differences in D, i.e. for $n \in \mathbb{N}$,

$$\delta_n(D) = \max\Big\{ \#W \subseteq \{0, \dots, K\}^n : w_1 - w_2 \neq udv \\ \text{for all } w_1, w_2 \in W,\ d \in D,\ u, v \in \{0, \dots, K\}^n \Big\}.$$

(*iv*) The *capacity* $\operatorname{cap}(D)$ of D is defined by

$$\operatorname{cap}(D) = \log_{K+1} \lim_{n\to\infty} \frac{\delta_n(D)}{n}. \tag{4.5.1}$$

(*v*) For sets of forbidden differences, we use the abbreviations $- = -1$, $\circ = 0$, $+ = 1$. Furthermore, we use the additional symbol $\pm$ meaning that whenever the symbol $\pm$ occurs, both occurrences of $+$ and $-$ are forbidden at that particular location.

In the following, we will always assume that $K = 1$.

For a given set of forbidden differences, the problem of computing its capacity (4.5.1) can be translated to the computation of the JSR of a finite set of matrices, see e.g. for details [Moision, Orlitsky and Siegel, 2001]. For $K = 1$, the occurring matrices in this application only have entries in $\{0, 1\}$, however, their dimension, as well as the number of matrices increases exponentially with the length of the forbidden difference words.

Example 4.5.8. For $D_3 = \{+\circ+\circ\}$, by [Moision, Orlitsky and Siegel, 2001], the capacity of D_3 is given by:

$$\operatorname{cap}(D_3) = \log_2 \operatorname{JSR}\Bigg(\Bigg\{ \begin{bmatrix} 0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&1&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&1&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&1&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&1\\0&0&0&1&0&0&0&0 \end{bmatrix}, \\ \begin{bmatrix} 0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&1 \end{bmatrix}, \\ \begin{bmatrix} 1&0&0&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&1&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&1\\0&0&0&1&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0 \end{bmatrix}, \\ \begin{bmatrix} 1&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0 \end{bmatrix} \Bigg\}\Bigg). \qquad \triangle$$

We use the modified invariant polytope algorithm 4.4.1 to compute the capacities for the forbidden differences from [Moision, Orlitsky and Siegel, 2001, page 10], [Blondel and Chang, 2011, Figure 1] and [Blondel, Jungers and Protasov, 2006, page 6 and Section v]. Nearly any of these capacities were known exactly before. The test results, together with the computed capacities are printed in Figure 4.12.

The difference set $D_4 = \{\circ\circ+\circ-\}$, taken from [Blondel and Chang, 2011, Figure 1], is an interesting test case for the modified Gripenberg algorithm, since the computation of the capacity translates to the JSR of a set with 256 matrices of dimension 16. As one can expect, the

D	*s.m.p.*	cap(D)	$\#V$	J	*dim*
$\pm\pm$	B_2B_3	1/2	$3 \cdot 2$	4	2
$\circ\pm$	B_3	0	$2 \cdot 2$	4	2
$\circ+-$	B_4B_1	0.6942...	$45 \cdot 2$	4	4
$\circ\pm\pm$	B_1B_2	1/2	$37 \cdot 2$	16	4
$\pm\pm\pm$	$B_6B_4B_1$	2/3	$19 \cdot 2$	16	4
$+-+-$	B_1B_2	0.9468...	$86 \cdot 2$	2	8
$+++-$	$B_1^3B_2^3$	0.9005...	$40 \cdot 2$	2	8
$++++$	B_1	0.9468...	$84 \cdot 2$	2	8
$\circ+-+$	B_3	0.8791...	$43 \cdot 2$	4	8
$\circ++-$	$B_1^2B_4^2$	0.8113...	$46 \cdot 2$	4	8
$\circ+++$	B_1	0.8791...	$46 \cdot 2$	4	8
$\circ++\pm$	$B_1^2B_2^2$	0.7396...	$244 \cdot 2$	16	8
$\circ+\circ+$	$B_4B_{11}^2B_{13}B_6^2$	0.7298...	$804 \cdot 2$	16	8
$\pm\pm\pm\pm$	$B_{86}B_{52}B_{16}B_1$	3/4	$357 \cdot 2$	256	8
$\circ+-+\circ$	$B_{11}B_{13}$	0.9163...	$1721 \cdot 2$	16	16
$\circ+++\circ$	B_4B_6	0.9163...	$4559 \cdot 2$	16	16
$+++++-\circ$	B_3	0.9761...	$992 \cdot 2$	4	64

Figure 4.12: Capacity of various difference sets D.
D: set of forbidden differences, cap(D): capacity of D, *dim*: dimension of the matrices, J: number of matrices, $\#V$: number of vertices of the invariant polytope, *s.m.p.*: an s.m.p..

Gripenberg algorithm fails to find an s.m.p., also the modified Gripenberg algorithm 4.3.5 fails. The genetic algorithm in most runs finds a better product than the one found by the Gripenberg algorithm. The modified invariant polytope algorithm 4.4.1 also finds this better product after a while, however, it does not terminate in reasonable time. Thus, the exact capacity, and whether an s.m.p. exists or not, is still unknown. The test results are in Figure 4.13.

Test set	*Algorithm*	*lower bd.*	*time*
$D_4 = \{\circ\circ+\ \circ-\}$	mod. invariant polytope	1.6736...	40 *s*
$J = 265$	mod. Gripenberg	1.6663...	2 *s*
$dim = 16$	random Gripenberg	1.6663...	2 *s*
	Gripenberg	1.6663...	60 *s*
	genetic	1.6736...	10 *s*

Figure 4.13: The modified Gripenberg algorithm 4.3.5 fails for the set of matrices corresponding to the forbidden difference set D_4 in Section 4.5.3.
dim: dimension of the matrices, *lower bd.*: computed lower bound for the JSR, J: number of matrices, *time*: time needed by the algorithm.

Remark 4.5.9. The exact computation of the capacity using the modified invariant polytope algorithm 4.4.1 is only possible if we use the estimates for the Minkowski norm in Lemma 4.4.6 (v), which reduces the number of norms to be computed by a factor of 100.

The function `codecapacity` [Mejstrik, 2018b] implements an algorithm by [Moision, Orlitsky and Siegel, 2001, Section IV], returning the set of matrices needed for the JSR computation of the capacity of a set of forbidden differences. It works for reasonably small difference sets, and theoretically for difference words with entries in $\{-K, \dots, K\}$, $K \in \mathbb{N}$. In practice it fails for $K \geq 3$, due to memory limitations.

4.5.4 Hölder exponents of Daubechies wavelets

An important application of the JSR is the computation of the regularity of *refinable functions*, as described in Chapter 3. We use the modified invariant polytope algorithm 4.4.1 to compute the Hölder regularity of the Daubechies wavelets D_n [Daubechies, 1988]. The regularity of D_2, D_3, and D_4 was computed by Daubechies and Lagarias [1992b], Gripenberg [1996] computed it for $D_5, \ldots, D_8$, then Guglielmi and Protasov [2016], as a demonstration of the invariant polytope algorithm, computed the regularities of $D_9, \ldots, D_{20}$. Now with the modified invariant polytope algorithm 4.4.1 we can compute the Hölder regularity for Daubechies wavelets up to D_{42}.

As noted in [Guglielmi and Protasov, 2016, Section 6.2], the polytopes generated by these matrices are very flat and the introduction of nearly-s.m.p.s and extra-vertices increases the performance of the invariant polytope algorithm 4.4.1 tremendously. In other words, using the wrong set of nearly-s.m.p.s, the algorithm does not terminate at all for some examples. These cases are marked with † in Figure 4.15. Working nearly-s.m.p.s and extra-vertices were merely found by trial and error.

We report the number of extra-vertices and the vertices of the cyclic roots from the nearly-s.m.p.s together under *#Extra-V*. The number of the vertices of the invariant polytopes is printed in Figure 4.14 (left side).

Remark 4.5.10. With the new regularity values for D_{21} to D_{42}, we can refine the observation in [Guglielmi and Protasov, 2015], that the differences of Hölder regularities $\alpha_n - \alpha_{n-1}$ seem to converge towards a value of 0.21 or maybe even 0.2, see Figure 4.14 (right side).

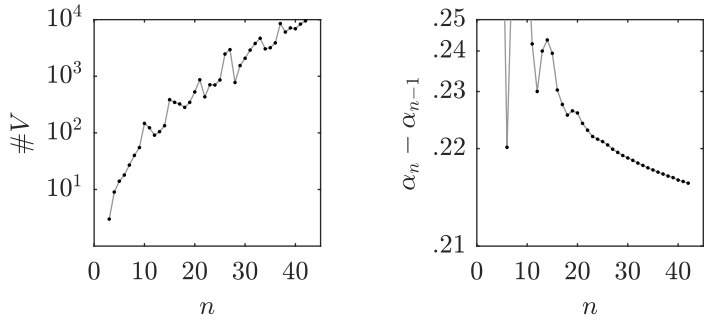


Figure 4.14: Left: Number of vertices of the invariant polytope $\#V$ against index n of Daubechies wavelet D_n. Right: Difference of regularities α of consecutive Daubechies wavelets D_n.

Remark 4.5.11. The function `daubechiesmask` [Mejstrik, 2018b] returns the mask coefficients of the Daubechies scaling functions. The function `daubechiesmatrix` [Mejstrik, 2018b] returns the transition matrices constructed from the mask coefficients, whose JSR determines the Hölder regularity of the Daubechies wavelets.

n	*s.m.p.*	*#Extra-V*	$\#V$	*time*	α
2	B_0	0	0·2	$< 5\,s$	0.55001...
3	B_0	0	3·2	$< 5\,s$	1.08783...
4	B_0	2	9·2	$< 5\,s$	1.61793...
5	B_0 and B_1	2	14·2	$< 5\,s$	1.96896...
6	B_0 and B_1	3	18·2	$< 5\,s$	2.18914...
7	B_0 and B_1	4	27·2	$< 5\,s$	2.46041...
8	B_0 and B_1	5	40·2	$< 5\,s$	2.76082...
9	B_0 and B_1	6	55·2	$< 5\,s$	3.07361...
10	$B_0^2B_1^2$	5	147·2	$< 5\,s$	3.36139...
11	B_0 and B_1	8	123·2	$7\,s$	3.60347...
12	B_0 and B_1	9	91·2	$7\,s$	3.83348...
13	B_0 and B_1	10	105·2	$6\,s$	4.07348...
14	B_0 and B_1	11	134·2	$8\,s$	4.31676...
15	$B_0^4B_1^2$	11	386·2	$6\,s$	4.55612...
16	$B_0^2B_1^2$	12	346·2	$7\,s$	4.78644...
17	B_0 and B_1	14	324·2	$5\,s$	5.01380...
18	B_0 and B_1	15	282·2	$8\,s$	5.23917...
19	B_0 and B_1	16	346·2	$9\,s$	5.46532...
20	B_0 and B_1	17	529·2	$12\,s$	5.69108...
21	$B_0^2B_1^2$	17	868·2	$15\,s$	5.91500...
22†	$B_0^2B_1^4$	22	433·2	$9\,s$	6.13779...
23	B_0 and B_1	20	707·2	$18\,s$	6.35958...
24	B_0 and B_1	21	701·2	$16\,s$	6.58096...
25	B_0 and B_1	22	861·2	$20\,s$	6.80198...
26	$B_0^4B_1^2$	22	2471·2	$73\,s$	7.02250...
27	$B_0^2B_1^2$	29	2952·2	$60\,s$	7.24241...
28†	$B_0^2B_1^6$	105	777·2	$24\,s$	7.46187...
29	B_0 and B_1	26	1545·2	$39\,s$	7.68091...
30	B_0 and B_1	27	2078·2	$64\,s$	7.89962...
31	B_0 and B_1	29	2898·2	$190\,s$	8.11801...
32	$B_0^2B_1^2$	29	3791·2	$760\,s$	8.33605...
33†	$B_0^2B_1^2$	30	4692·2	$1330\,s$	8.55379...
34	B_0 and B_1	32	3047·2	$628\,s$	8.77123...
35	B_0 and B_1	33	3191·2	$727\,s$	8.98841...
36	B_0 and B_1	34	3887·2	$881\,s$	9.20533...
37	$B_0^6B_1^2$	70	8529·2	$6503\,s$	9.42202...
38	$B_0^2B_1^2$	38	6035·2	$3540\,s$	9.63847...
39	$B_0^2B_1^4$	40	7142·2	$3900\,s$	9.85474...
40	B_0 and B_1	38	6909·2	$5550\,s$	10.07073...
41	B_0 and B_1	39	8343·2	$8743\,s$	10.28656...
42	B_0 and B_1	40	9508·2	$16373\,s$	10.50220...

Figure 4.15: Hölder regularity of Daubechies wavelets.
α: Hölder regularity of the Daubechies wavelet D_n, n: index of the Daubechies wavelet D_n, $\#V$: number of vertices of the invariant polytope, *#Extra-V*: number of extra-vertices including those from nearly-s.m.p.s., *s.m.p.*: an s.m.p., *time*: time needed to compute the invariant polytope.
†: See explanation in Section 4.5.4.

4.6 Appendix for Chapter 4

Proof for Example 4.5.3. Define $\tilde{C}_n = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, $n > 0$. A product of $C_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\tilde{C}_n$ is non-zero if and only if it is of the form $C_0^{l_1}\tilde{C}_n C_0^{l_2}\tilde{C}_n \cdots \tilde{C}_n C_0^{l_M}$, $l_m \in \mathbb{N}$, $m \in \{1,\dots,M\}$. Since the spectral radius does not change under cyclic permutations, we can assume that the product is of the form $C_0^{l_1}\tilde{C}_n C_0^{l_2}\tilde{C}_n \cdots C_0^{l_M}\tilde{C}_n$, $l_m \in \mathbb{N}$, $m \in \{1,\dots,M\}$, $M \in \mathbb{N}$. Note that for $M \geq 2$ we can assume that

$$l_m \geq 2 \quad \text{for at least one } m \in \{1,\dots,M\}, \tag{4.6.1}$$

since otherwise $l_m = 1$ for all $m \in \{1,\dots,M\}$ and thus we could reduce the product to the form $C_0^{l_1}\tilde{C}_n$. A straightforward computation gives for $l_m \in \mathbb{N}$, $m \in \{1,\dots,M\}$, that $C_0^{l_m} = \begin{bmatrix} 1 & l_m \\ 0 & 1 \end{bmatrix}$, and thus $C_0^{l_m}\tilde{C}_n = n\begin{bmatrix} l_m & 0 \\ 1 & 0 \end{bmatrix}$, from which it follows that

$$C_0^{l_1}\tilde{C}_n \cdots C_0^{l_M}\tilde{C}_n = n^M \begin{bmatrix} l_1 \cdots l_M & 0 \\ l_2 \cdots l_M & 0 \end{bmatrix}.$$

Since the length of this product is equal to $l_1 + \cdots l_M + M$ the averaged spectral radius of this product is $(n^M l_1 \cdots l_M)^{1/(M+l_1+\cdots l_M)}$. We define $P = n^M l_1 \cdots l_M$ and $S = M + l_1 + \cdots l_M$. Note that $P \neq 1$ by (4.6.1). Taking the total derivative D of $P^{1/S}$ with respect to $(l_1,\dots,l_M)$ and setting it to zero we obtain

$$D(P^{1/S}) = \frac{P^{1/S}}{S^2}\begin{bmatrix} l_1^{-1}(S - l_1 \log P) \\ \vdots \\ l_M^{-1}(S - l_M \log P) \end{bmatrix}^T = 0.$$

Thus, $S = l_m \log P$ or, equivalently, $S/\log P = l_m$, $m \in \{1,\dots,M\}$. It follows that all l_m, $m \in \{1,\dots,M\}$, must be equal. Therefore, the averaged spectral radius of all finite products is maximized with a product of the form $C_0^l\tilde{C}_n$, $l \in \mathbb{N}$, whose averaged spectral radius is equal to $nl^{1/(1+l)}$. For fixed $l \in \mathbb{N}$ this term has its maximum at

$$\frac{l^{1/(1+l)}n(1 + l - l\log nl)}{l(1+l)^2} = 0$$

which yields $n = \frac{1}{l}e^{(1+l)/l}$. Thus, $C_0^l\tilde{C}_n$ is the product with largest averaged spectral radius under all finite products or, equivalently, with $C_n = \begin{bmatrix} 0 & 0 \\ \frac{1}{n}e^{1+\frac{1}{n}} & 0 \end{bmatrix}$, $n \in \mathbb{N}$, the product $C_0^n C_n$ has largest averaged spectral radius under all finite products with matrices from the set $\mathcal{C} = \{C_0, C_n\}$. Using (4.1.3) we conclude that $\mathrm{JSR}(\mathcal{C}) = \rho(C_0^n C_n)^{1/(n+1)} = (e^{(n+1)/n})^{1/(n+1)} = e^{1/n}$. □

Theorem 4.6.2 (Fekete's Lemma, [Fekete, 1923])**.** *If $(a_n)_{n\in\mathbb{N}} \in \ell(\mathbb{N})$ is sub-additive, i.e.*

$$a_{n+m} \leq a_n + a_m \quad \textit{for all} \quad n, m \in \mathbb{N}, \tag{4.6.2}$$

then

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}.$$

Proof. Let

$$A = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$$

and $B \in \mathbb{R}$ such that $B > A$. Choose $b \in \mathbb{N}$ such that $\frac{a_b}{b} < B$. For all $n \in \mathbb{N}$, $n \geq k$ there exist $p_n, q_n \in \mathbb{N}$ such that $n = p_n b + q_n$ and $0 \leq q_n \leq b-1$. By applying (4.6.2) we obtain $a_n = a_{p_n b + q_n} \leq a_{p_n b} + a_{q_n} \leq p_n a_b + a_{q_n}$, and thus

$$\frac{a_n}{n} \leq \frac{b p_n}{n} \frac{a_b}{b} + \frac{a_{q_n}}{n},$$

When n goes to infinity, $\frac{p_n b}{n}$ converges to 1 and $\frac{a_{q_n}}{n}$ converges to 0. Therefore, we have for all $B > A$

$$A \leq \lim_{n\to\infty} \frac{a_n}{n} \leq \frac{a_b}{b} < B.$$

Finally, letting B going to A we obtain $A = \inf_{n\in\mathbb{N}} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n}{n}$. □

Fekete's Lemma also applies to sub-multiplicative sequences.

Corollary 4.6.3. *If $(a_n)_{n\in\mathbb{N}} \in \ell(\mathbb{N}, \mathbb{R}_+)$ is sub-multiplicative, i.e.*

$$a_{n+m} \leq a_n a_m \quad \textit{for all} \quad n, m \geq 1,$$

then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \inf_{n\in\mathbb{N}} \sqrt[n]{a_n}.$$

Proof. If $a_N = 0$ for any $N \in \mathbb{N}$, then $a_n = 0$ for all $n \geq N$, and thus $\lim_{n\to\infty} \sqrt[n]{a_n} = \inf_{n\in\mathbb{N}} \sqrt[n]{a_n} = 0$.

So, we suppose $a_n > 0$ for all $n \in \mathbb{N}$. Since $(\log a_n)_{n\in\mathbb{N}}$ is a sub-additive sequence, by Theorem (4.6.2), $\inf_{n\in\mathbb{N}} \log\left(a_n^{1/n}\right) = \lim_{n\to\infty} \log\left(a_n^{1/n}\right)$. Using that the logarithm is continuous, the claim follows. □

Algorithm 4.6.1. Simple implementation of the modified Gripenberg algorithm 4.3.5 in Matlab.

```
function [ c ] = gripenberg_modified( M, N, D )
%Tries to find smp-candidates in a fast way.
%   M      input matrices
%   N      number of products kept in each step
%   D      maximal length of products to be computed
%Ex: gripenberg_modified( { [ 2 1; 0 -2 ], [ 2 1; -1 -2 ] }, 4, 10 )
J = length(M);                  %number of matrices
o = 1:J;                        %the orderings of the products to be checked
c = {};                         %list of candidates
r = 0;                          %lower bound for JSR
for d = 1:D                     %do D iterations
    NR = zeros(2,size(o,2));    %norm and rho of candidates
    for i = 1:size(o,2)         %can be parallelised using parfor!
        P = buildProduct(M,o(:,i)); %construct matrices
        NR(:,i) = [ norm(P); max(abs(eig(P))) ];  %compute norm and rho
    end
    NR = NR.^(1/d);             %average norm and rho
    if r < max(NR(2,:))         %test if new bound was found
        c = {};                 %delete candidates
        r = max(NR(2,:));       %update lower bound for JSR
    end
    c = [ c num2cell(o(:,NR(2,:) >= r),1) ]; %add candidates
    idx = NR(1,:) < r;          %remove products with norm less than JSR
    NR(:,idx) = [];
    o(:,idx) = [];
    [NR,idx] = sortrows(NR');   %sort correspdonding to norm
    NR = NR.'; idx = idx.'; nNR = size(NR,2);
    if nNR > 2*N                %keep highest and lowest norms
        o = o(:,[ idx(1:N) idx(nNR-N+1:nNR) ]);
    else                        %keep everything if N is too big
        o = o(:,idx);
    end
    o = [repmat(o,[ 1 J ]);       %make new orderings of products
    reshape(repmat(1:J,[ size(o,2) 1 ]),1,[])];
end

function [ M ] = buildProduct( A, prod )
% Constructs the product of matrices of A corresponding to prod.
M = eye(size(A{1},1));
for t = 1:length(prod);
    M = A{prod(t)}*M;
end
```

5 Appendix

Theorem 5.0.1 proofs Conjecture 3.3.4.

Theorem 5.0.1. *If $\mathcal{S}^{\mathbb{N}}$ is convergent, then there exists a unique, with respect to inclusion, minimal set $\Omega \subseteq \mathbb{Z}^s$ such that $\ell(\Omega)$ is $\mathcal{T}$ invariant.*

Proof. Assume the contrary. Then there exist sets $\Omega_1, \Omega_2 \subseteq \mathbb{Z}^s$, finite, such that $\ell(\Omega_1), \ell(\Omega_2)$ are $\mathcal{T}$-invariant. In particular, the set $\Omega = \Omega_1 \cup \Omega_2$ is $\mathcal{T}$-invariant and the transition matrices $T_{d,j,\Omega}$, $d \in D_j$, $j \in \{1, \dots, J\}$, have the form

$$T_{d,j,\Omega} = \begin{bmatrix} T_{d,j,\Omega_1} & 0 \\ 0 & T_{d,j,\Omega_2} \end{bmatrix}.$$

Both matrices $T_{d,j,\Omega_1}, T_{d,j,\Omega_2}$ have an eigenvalue 1. Thus, $T_{d,j,\Omega}$ has an eigenvalue 1 whose corresponding eigenvector is not simple. But, by the uniqueness of the basic limit functions, Proposition 3.1.13 (vi), the eigenvector to the eigenvalue 1 of the transition matrices of convergent subdivision schemes is unique, which is a contradiction. □

Source code

The source code of all mentioned programs may be found

- at the personal homepage of the author: **tommsch.com/science.php**, or
- at the Matlab File Exchange: **mathworks.com/matlabcentral/fileexchange**, or
- at the thesis-repository of the University of Vienna: **othes.univie.ac.at**, or
- comes together with this thesis on this page.

Glossary

Symbols

$\lvert\cdot\rvert$	Absolute value → Multi index notation 12,
$*$	Conjugate transpose, $M^* = \bar{M}^T$
∂	Boundary of a set 12,
$[\cdot]$	Denotes sequences, matrices or intervals 13,
$\emptyset$	Empty set 12,
$\triangle$	End of an example 13,
!	Factorial 12,
$\lfloor\cdot\rfloor$	Floor function 13,
$\#$	Cardinality of a set 12,
$(\cdot,\cdot)$	Inner product, $(x,y) = \sum_{\alpha\in\mathbb{Z}^s} x(\alpha)y(\alpha)$ 13,
$^\circ$	Interior of a set 12,
$\lVert\cdot\rVert^{1/k}$	Averaged norm with respect to the length of a matrix product 61,
$\lVert\cdot\rVert_{\ell_p}$, $\lVert\cdot\rVert_p$	ℓ_p-Norm 13,
$\lVert\cdot\rvert_{\nabla^{k+1}}\rVert_\infty$	Restricted norm 45,
$\lVert\cdot\rVert_P$	Minkowski norm with respect to a set $P \subseteq \mathbb{R}^s$ 64,
$-$, $\circ$, $+$, $\pm$	Numbers -1, 0, $+1$, ± 1 used in difference words 89,
$+$, $-$, $\pm$	Usual operators for numbers or the pairwise sum of sets 12,
$\simeq$	Either isomorphic (for spaces), *essentially equal* (for sets with respect to the Lebesgue measure) or *approximately equal* (for numbers) 13,
$\subseteq$, $\subsetneq$, $\not\subseteq$	Subset or equal, Strict subset, Not a subset 12,
Y^X	Sequence space 13,
4	∞ 27,
a	Usually the mask of a subdivision operator, $a \in \ell_0(\mathbb{Z}^s)$ 27,
$a^*(z)$	Symbol, the Laurent polynomial corresponding to a sequence a, $a^*(z) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)z^\alpha$, $z \in \mathbb{C}^s \setminus \{0\}$ 49,
a'	Usually the mask of a difference subdivision operator, $a' \in \ell_0(\mathbb{Z}^s, \mathbb{R}^{\binom{s+k}{s-1}})$ 42,
absco	Absolutely convex hull 64,
$\mathcal{A}^n$	Set of matrix products of length n with matrices from $\mathcal{A}$ 12,
b_{k+1}	Intermediate bound for the JSR 81,
$\mathbb{C}$	Complex numbers 12,
C^α	Hölder continuous functions with Hölder regularity α 13,
c	Usually the starting sequence for a subdivision scheme, $c \in \ell(\mathbb{Z}^s)$ 28,
cap	Capacity of a difference set 88,
cl	Closure of a set 12,
co, co_-, co_s	Convex hull, Cone hull with respect to the first orthant, Symmetrized convex hull 64,
co_*	Any of the following: co, co_s, absco, co_- 64,
D	Usually a digit set 17,
d	Usually an element from a digit set

$\mathbb{D}$	The double dominant double dragon 17,
$.d_1d_2\ldots$	Expansion of a number with digits d_n 16,
$g_{c,\boldsymbol{j}}$, g_c	Limit function of a subdivision scheme 28,
e_l	The l^{th} standard unit vector of $\mathbb{R}^s$ 13,
h	Hausdorff metric, $\mathrm{h}(B,C) = \max\{\sup_{b\in B}\inf_{c\in C}\|b-c\|_2\,,\,\sup_{c\in C}\inf_{b\in B}\|b-c\|_2\}$ 16,
$\mathcal{H}$	Space of all non-empty, compact subsets of K, $K\subseteq\mathbb{R}^s$, with the Hausdorff metric h 16,
$\mathcal{H}$	Cyclic root for the (modified) invariant polytope algorithm 67,
I	Identity matrix 13,
J	Usually the number of matrices M_j, subdivision operators S_j or matrices A_j. 16, 27, 59,
$\boldsymbol{j}$	Index sequence, usually for the operators in the set $\mathcal{S}$ 28,
$\boldsymbol{j}^{[r]}$	Shifted sequence $\boldsymbol{j}^{[r]} = (j_r, j_{r+1}, j_{r+2}, \cdots)$ 20,
JSR	Joint spectral radius of a set of matrices 15,
k	Usually the order of sumrules or the degree of a polynomial 29, 41, 42, 46,
$K_{Q,\boldsymbol{j}}$	Attractor corresponding to dilation matrices $M_j\in\mathbb{Z}^{s\times s}$, finite sets $Q_j\subseteq\mathbb{Z}^s$ and ordering $\boldsymbol{j}\in\{1,\ldots,J\}^{\mathbb{N}}$ 16,
l	Usually the running index of the dimension, $l\in\{1,\ldots,s\}$ 13,
$l(\,\cdot\,)$	Length of a matrix product 61,
ℓ	Sequence space 13, 35,
M	Usually the dilation matrix used for subdivision, $\rho(M^{-1})<1$ 27,
$\mathcal{M}_\eta$	Sequences whose corresponding matrix products have uniformly averaged spectral radii less than η 82,
$\mathbb{N}$	The positive integers excluding zero 12,
$\mathbb{N}_0$	The positive integers including zero 12,
$\mathbb{Q}$	The rational numbers 12,
$.q_1q_2\ldots$	Expansion of a number with digits q_n 16,
r	Usually the starting index in a subdivision scheme, $r\in\mathbb{N}$ 32,
$\mathbb{R}_+$	The positive real numbers including zero 12,
$\mathbb{R}$	The real numbers 12,
$\mathbb{R}^{I\times J}$, $\mathbb{R}^\Omega$	Generalized matrix notation 13,
RSR	Restricted spectral radius of a set of difference subdivision operators 45,
S	Subdivision operator, $S=(a,M)$, $Sc=\sum_{\beta\in\mathbb{Z}^s}a(\,\cdot\,-M\beta)c(\beta)$ 27,
s	Usually the dimension 13,
S'	Difference subdivision operator $S'=(a',M)$ to the corresponding subdivision operator $S=(a,M)$ 42,
$(S_{j_n})_{n\in\mathbb{N}}$	Subdivision scheme, $(S_{j_n})_n\in\mathcal{S}^{\mathbb{N}}$ 27,
s.m.p.	Spectral maximizing product 61,
span	Linear span of a finite set of vectors 13,
$\mathcal{S}'$	Finite set of difference subdivision operators corresponding to the subdivision operators in $\mathcal{S}$ 42,
$\mathcal{S}$	Finite set of subdivision operators 27,
supp	Support of a sequence 13,
$\mathcal{T}_{d,j}$	Transition operator corresponding to the subdivision scheme $S_j=(a_j,M_j)$ and digit $d\in D_j\simeq\mathbb{Z}^s/M\mathbb{Z}^s$, $\mathcal{T}_{d,j}c=\sum_{\beta\in\mathbb{Z}^s}a_j(M_j\,\cdot\,-\beta+d)c(\beta)$, $c\in\ell(\mathbb{Z}^s)$ 34,

$\mathcal{T}$	Set of transition operators 34,
$T_{d,j,\Omega}$	Transition matrix corresponding to the subdivision scheme S_j and digit $d \in D_j$ restricted to the space $\ell(\Omega)$, $T_{d,j,\Omega} = \big[a_j(M_j\alpha - \beta + d)\big]_{\alpha,\beta\in\Omega}$ 36,
V_k	Space orthogonal to Π_k, $V_k(\Omega) = \{v \in \ell(\Omega) : \sum_{\beta\in\Omega} v(\beta)p(-\beta) = 0 \text{ for all } p \in \Pi_k\}$ 46,
$\tilde{V}_k$	Space spanned by difference sequences, $\tilde{V}_k(\Omega) = \operatorname{span}\{\tilde{\nabla}^\mu \delta(\,\cdot - \beta) \in \ell(\mathbb{Z}^s) : \beta \in \mathbb{Z}^s,\ \mu \in \mathbb{N}_0^s,\ \lvert\mu\rvert = k+1,\ \operatorname{supp} v \subseteq \Omega\}$ 46,
X_μ	Index set for difference sequences 47,
$\mathbb{Z}$	The integers 12,
z^M	$\to$ Multi index notation 12,
α, β	Usually indices in $\mathbb{Z}^s$ 13,
δ	Usually the Kronecker delta 13,
δ_n	Biggest cardinality of a set which avoids forbidden differences 88,
$\lambda(\,\cdot\,)$	Lebesgue-measure of a set 12,
μ	Usually used for the superscript in the backward difference operator $\tilde{\nabla}^\mu$ 41,
μ, ν	Usually used as multi-indices 12,
∇	Backward difference operator, $\nabla_l c = c - c(\,\cdot\, - e_l)$, $\nabla = \begin{bmatrix}\nabla_1 & \nabla_2 & \dots & \nabla_s\end{bmatrix}^T$, $\tilde{\nabla}^\mu = \prod_{l=1}^s (\nabla_l)^{\mu_l}$ with $\mu \in \mathbb{N}_0^s$, ∇^{k+1} is the column vector of all possible backward differences $\tilde{\nabla}^\mu$ with $\lvert\mu\rvert = k+1$ 41,
Ω_C	Minimal $\mathcal{T}_{d,j}$ invariant set containing $0 \in \mathbb{Z}^s$ 36,
$\Omega_\mathbb{R}$	Supertile corresponding to the sets $\operatorname{supp} a_j - D_j$, $j \in \{1,\dots,J\}$ 35,
Ω_V	$\mathcal{T}_{d,j}$ invariant set such that $V_k(\Omega) = \tilde{V}_k(\Omega)$ for all $k \in \mathbb{N}_0$, $X_\mu \neq \emptyset$ for all $\lvert\mu\rvert = k+1$ 36,
$\Omega_\mathbb{Z}$	$\Omega_\mathbb{R} \cap \mathbb{Z}^s$ 35,
Ω	Usually $\mathcal{T}_{d,j}$ invariant set 36,
$\phi_{\boldsymbol{j}}$	Basic limit function, $\phi_{\boldsymbol{j}^{[r]}} = \sum_{\alpha\in\mathbb{Z}^s} a_{j_r}(\alpha)\phi_{\boldsymbol{j}^{[r+1]}}(M_{j_r}\,\cdot - \alpha)$ 30,
Π_k	Space of polynomial sequences of degree $k \in \mathbb{N}_0^s$ 42,
ρ	Spectral radius of a matrix 13,
$\rho(\,\cdot\,)^{1/k}$	Averaged spectral radius of a matrix product of length $k \in \mathbb{N}$ 61,

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Abstract

This thesis extends the matrix based approach to the setting of multiple subdivision schemes studied in [Sauer, 2012]. Multiple subdivision schemes, in contrast to stationary and non-stationary schemes, allow for level dependent subdivision weights and for level dependent choice of the dilation matrices. The latter property of multiple subdivision makes the standard definition of the transition matrices, crucial ingredient of the matrix approach in the stationary and non-stationary settings, inapplicable. We show how to avoid this obstacle and characterize the convergence of multiple subdivision schemes in terms of the joint spectral radius of certain square matrices derived from subdivision weights.

Albeit the characterization of the convergence of multiple subdivision schemes in terms of the joint spectral radius is elegant, the numerical computation of the joint spectral radius still poses big problems. In several papers of 2013 – 2016, Guglielmi and Protasov made a breakthrough in the problem of the joint spectral radius computation, developing the invariant polytope algorithm, which for most matrix families finds the exact value of the joint spectral radius. This algorithm found many applications in problems of functional analysis, approximation theory, combinatorics, etc.. In this thesis we propose a modification of the invariant polytope algorithm making it roughly three times faster and suitable for higher dimensions. The modified version works for most matrix families of dimensions up to 25, for non-negative matrices the dimension is up to 3000.

Besides, we introduce a new, fast algorithm for computing good lower bounds for the joint spectral radius, which finds in most cases the exact value of the joint spectral radius in less than 5 seconds. Corresponding examples and statistics of numerical results are provided.

Zusammenfassung

Die vorliegende Arbeit verallgemeinert die *Matrix-Methode* für die Konvergenzanalyse von Unterteilungsalgorithmen auf *multiple Unterteilungsalgorithmen*, wie sie von Sauer [2012] eingeführt wurden. Multiple Unterteilungsalgorithmen erlauben, anders als stationäre und nicht-stationäre Unterteilungsalgorithmen, level-unabhängige Verfeinerungsgewichte und Verfeinerungsmatrizen. Letztere Eigenschaft erfordert eine neue Definition der Übergangsmatrizen, welche ein grundlegendes Objekt der Matrix Methode darstellen. Wir zeigen wie Übergangsmatrizen im multiplen Rahmen konstruiert werden und charakterisieren über den gemeinsamen Spektralradius dieser Matrizen die Konvergenz von multiplen Unterteilungsalgorithmen.

Die Charakterisierung der Konvergenz von multiplen Unterteilungsalgorithmen mittels Übergangsmatrizen ist zwar elegant, die numerische Berechnung des gemeinsamen Spektralradius birgt jedoch Schwierigkeiten. Der *invariante Polytop Algorithmus* von Guglielmi und Protasov (2013 – 2016) stellt einen Durchbruch in der Berechnung des gemeinsamen Spektralradius dar. Der invariante Polytop Algorithmus findet für eine große Klasse von Matrixfamilien den genauen Wert ihres gemeinsamen Spektralradius. Der Algorithmus fand Anwendung bei der Lösung diverser Probleme, unter anderem in der Funktionalanalysis, Approximationstheorie und Kombinatorik. In dieser Arbeit schlagen wir Modifikationen des invarianten Polytop Algorithmus vor, um ihn schneller, robuster und für Matrizen höherer Dimensionen tauglich zu machen. Der modifizierte Algorithmus berechnet den exakten gemeinsamen Spektralradius für die meisten Matrixfamilien bis zu Dimension 25, für nicht-negative Matrizen bis zu Dimension 3000.

Weiters stellen wir einen neuen Algorithmus, genannt *modifizierter Gripenberg Algorithmus*, vor, der sehr gute untere Schranken für den gemeinsamen Spektralradius von Matrizen in weniger als fünf Sekunden berechnet. In den meisten unserer Tests fand der modifizierte Gripenberg Algorithmus sogar den genauen Wert des gemeinsamen Spektralradius. Wir demonstrieren die numerische Effizienz der Algorithmen an zahlreichen Testbeispielen.