Joint spectral radius algorithms and applications

2024-05-07

Charina, *Mejstrik*, Protasov, Reif

- Algorithms to compute the JSR
- Applications
- Properties
- Generalizations
- Shortfalls

Notation

Definition (*JSR*, Gian-Carlo Rota, Gilbert Strang, 1960)

$$\mathrm{JSR}(\mathcal{A}) := \lim_{n\to\infty} \max_{A_j\in\mathcal{A}} ||A_{j_n}\cdots A_{j_1}||^{1/n}$$

$\mathcal{A} \subseteq \mathbb{R}^{s\times s}$ is compact.

Continuity of the JSR

Theorem ([RS60, Bar88, BW92, Wir02, Koz10, EW24...])

The JSR *is continuous/locally Lipschitz/... w.r.t the Hausdorff metric.*

Corollary

Numerical computation makes sense.

Irreducibility

Definition (Irreducibility)

- If there exists $V \in \mathbb{R}^{s\times s}$ such that $VA_jV^{-1} = \begin{bmatrix} B_j & C_j \\ 0 & D_j \end{bmatrix}$ for all $A_j \in \mathcal{A}$, then $\mathcal{A}$ is reducible.

In other words: There exists no common invariant subspace (except $\{0\}$ and $\mathbb{R}^s$).

Theorem

If $\mathcal{A}$ is reducible, then $\mathrm{JSR}(\mathcal{A}) = \max\{\mathrm{JSR}(\mathcal{B}), \mathrm{JSR}(\mathcal{D})\}$.

Computing the JSR

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- Lower bounds (*Modified Gripenberg [M]*, *Genetic* [Chang])

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- Approximate computations
- Exact computations
  - **I**nvariant **p**olytope **a**lgorithm [Guglielmi, M., Protasov, Wirth, Zennaro, . . .] ( )
  - **F**inite **e**xpressible **t**ree **a**lgorithm [Möller, Reif] ( )

ttoolboxes

- Matlab toolbox for computation of the joint spectral radius
- gitlab.com/tommsch/ttoolboxes
- Mathematically rigorous results
- Brown functions are written by me
- Plots done using plotm
- ttoolboxes are extensively tested using TTEST

```
T = {[1 1;2 1], [1 0;-1 1]};
ipa( T );
```

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- The JSR toolbox [Vankeerberghen, Hendrickx, Jungers]
- Implementation of approximating algorithms
- At Matlab File Exchange

Paid advertisement

Gripenberg algorithms

Computation: Gripenberg algorithms

Gripenberg algorithm [Gripenberg, 1996] based on [DL],

Theorem ([DL])

$$\max_{A_j \in \mathcal{A}} \rho\left(A_{j_k} \cdots A_{j_1}\right)^{1/k} \leq \mathrm{JSR}(\mathcal{A}) \leq \max_{A_j \in \mathcal{A}} \left|\left|A_{j_k} \cdots A_{j_1}\right|\right|^{1/k}.$$

• Sorts out matrix products, which are proven to have averaged spectral radius less than the joint spectral radius of the given set.

Finiteness property [LW95, BM02, HST10]

Definition (s.m.p.)

- If there exists $\Pi = A_{j_N} \cdots A_{j_1}$ such that $\rho(\Pi)^{1/N} = \mathrm{JSR}(\mathcal{A})$, the set $\mathcal{A}$ possesses the *finiteness property*.
- Π is called an *spectral maximizing product (s.m.p.)*.

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Example

$$A_1 = \begin{bmatrix} 15/92 & -73/79 \\ 56/59 & 89/118 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -231/241 & -143/219 \\ 103/153 & -38/65 \end{bmatrix}$$

$\mathcal{A}$ has s.m.p. of length 119.

Joint spectral radius

- Computation of the JSR is NP-hard [Blondel, Tsitsiklis, 97].

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- JSR ≤ 1 is undecidable [Blondel, Tsitsiklis, 00].
- $\{\mathcal{A} : \text{JSR}(\mathcal{A}) \leq 1\}$ is non-algebraic [Kozyakin].

Computation: Gripenberg algorithms

Modified Gripenberg algorithm [M.]

- Sorts out matrix products according to a very stupid rule.

Examples: Gripenberg algorithms

$\mathcal{A} = \{A_j \in \mathbb{R}^{8\times 8} : \text{random entries}, \, j = 1, \ldots, 8\}$
100 test runs

Algorithm	time	success
mod. Gripenberg	$5.4\,s$	100%
Gripenberg	$82.1\,s$	100%
genetic	$12.0\,s$	87%
brute force	$180.0\,s$	74%

success: success rate of finding a product which attains the JSR.
time: runtime

Examples: Gripenberg algorithms

```
T = gallery_matrixset( 'J',2, 'dim',2, 'rho',1 )
c = findsmp( T )
```

Invariant polytope algorithms

Invariant polytope algorithms

Theorem (DL)

$$\max_{A_j \in \mathcal{A}} \rho\left(A_{j_k} \cdots A_{j_1}\right)^{1/k} \leq \mathrm{JSR}(\mathcal{A}) \leq \max_{A_j \in \mathcal{A}} \left|\left|A_{j_k} \cdots A_{j_1}\right|\right|^{1/k}.$$

Corollary

$$\mathrm{JSR}(\mathcal{A}) \leq \max_{A_j \in \mathcal{A}} \left|\left|A_j\right|\right|$$

The Pody

Theorem ([BW92])

Given a bounded set of matrices $\mathcal{A} \subseteq \mathbb{R}^{s\times s}$*. For* $v \in \mathbb{R}^s \setminus \{0\}$ *define*

$$P(v) = \mathrm{co} \bigcup_{A_j \in \mathcal{A},\ n \in \mathbb{N}_0} \{\pm A_{j_n} \dots A_{j_1} v\}.$$

1. *If* $\mathrm{JSR}(\mathcal{A}) \geq 1$ *and for some* $v \in \mathbb{R}^s$ *the set* $P(v)$ *is bounded and has non-empty interior, then* $\mathcal{A}$ *is irreducible and* $\mathrm{JSR}(\mathcal{A}) = 1$*.*
2. *If* $\mathcal{A}$ *is irreducible, and* $\mathrm{JSR}(\mathcal{A}) = 1$*, then* $P(v)$ *is a non-empty, bounded subset of* $\mathbb{R}^s$ *for any* $v \in \mathbb{R}^s$*.*

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2. *If $\mathcal{A}$ is irreducible, and $\mathrm{JSR}(\mathcal{A}) = 1$, then $P(v)$ is a non-empty, bounded subset of $\mathbb{R}^s$ for any $v \in \mathbb{R}^s$.*

→ $P(v)$ is the unit ball of a norm

→ “First” ipa [GZ08]: $P(v)$ has ∞ vertices

→ ipa [GP13]: Use leading eigenvector of s.m.p.

🍺: How does it work

$$A_1 = \lambda^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \lambda^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad \lambda = 4\sqrt{3} + 7$$

$$\mathcal{A} = \{A_1, A_2\}$$

$$\Pi = A_2 A_1 A_2^2 A_1 A_2 A_1^2, \quad \rho(\Pi) = 1$$

$$v = \begin{bmatrix} \sqrt{3} + 1 & 1 \end{bmatrix}^T, \quad w = \begin{bmatrix} (\sqrt{3} + 1)/2 & 1 \end{bmatrix}$$

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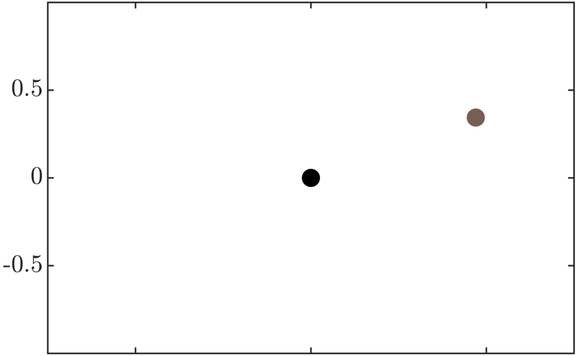

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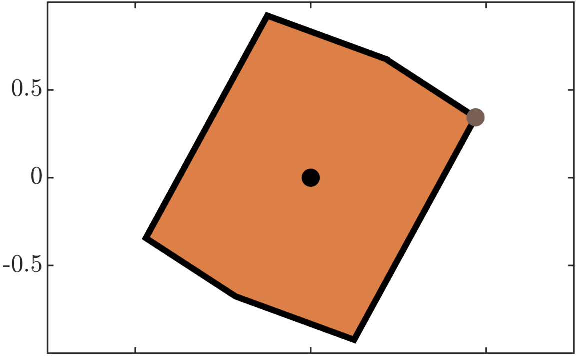

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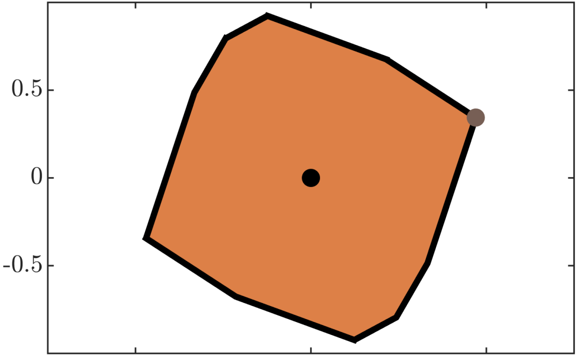

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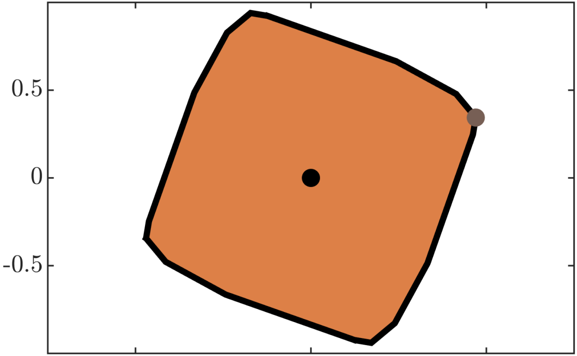

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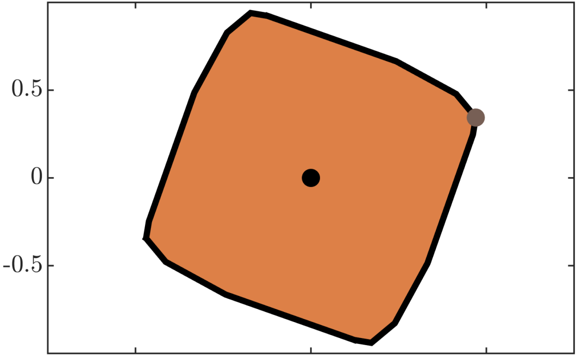

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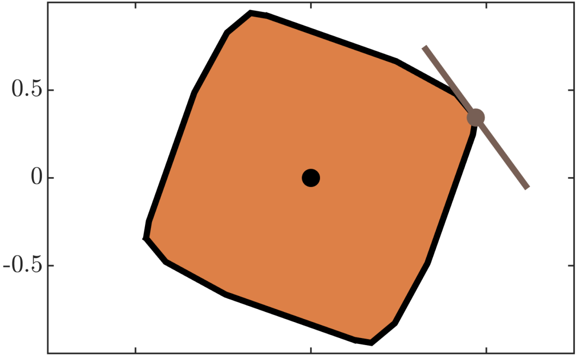

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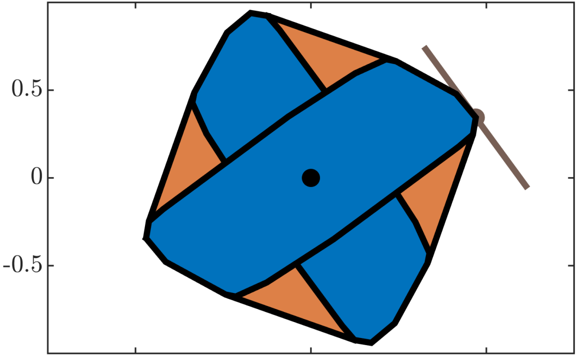

🍺

Theorem (🍺 [GP13,16])

Given finite $\mathcal{A} \subseteq \mathbb{R}^{s\times s}$, $\mathrm{JSR}(\mathcal{A}) \geq 1$, and one s.m.p. Π. The following are equivalent:

- *$\mathcal{A}$ has a spectral gap, there exist "no other" s.m.p.s, and Π has a unique leading eigenvalue λ.*
- *The ipa terminates.*

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Given finite $\mathcal{A} \subseteq \mathbb{R}^{s\times s}$, $\mathrm{JSR}(\mathcal{A}) \geq 1$, and one s.m.p. Π. The following are equivalent:
• $\mathcal{A}$ has a spectral gap, there exist "no other" s.m.p.s, and Π has a unique leading eigenvalue λ.
• The ipa terminates.

- Spectral gap is needed for equivalence and rigorousness.
- Can be generalized to multiple s.m.p.s.

Spectral gap

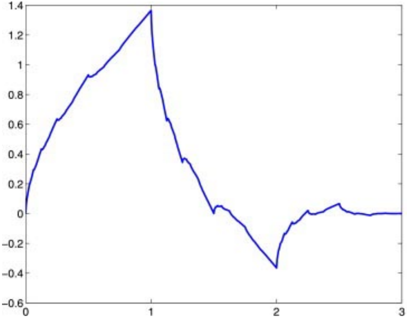
Definition

Given bounded $\mathcal{A} \subseteq \mathbb{R}^{s \times s}$ with $\lambda = \mathrm{JSR}(\mathcal{A}) > 0$, and let $\tilde{A} = \{A_j/\lambda : A_j \in \mathcal{A}\}$.
$\mathcal{A}$ has a spectral gap if for every product $\tilde{\Pi} = \tilde{A}_{j_n} \cdots \tilde{A}_{j_1}$, $\tilde{A}_j \in \tilde{\mathcal{A}}$, which is not an s.m.p., there exists $\gamma < 1$ such that $\rho(\tilde{\Pi}) < \gamma$.

Example

$\mathcal{A} = \{1,\ 1/2\}$ has a spectral gap.

Mutually refinable functions



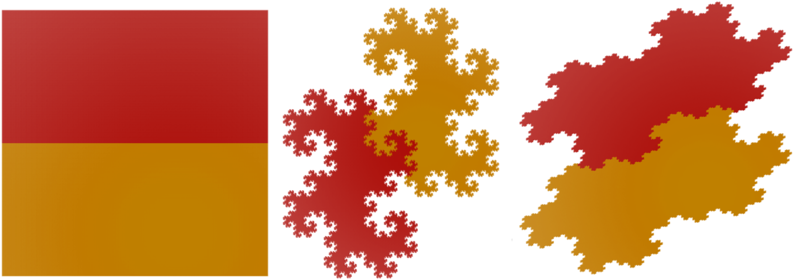
Mutually refinable functions

• Given mask $a \in \ell_0(\mathbb{Z})$. Does there exist a non-zero function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) = \sum_{\beta \in \mathbb{Z}} a(\beta)\phi(2x - \beta)$$

• Such a function exists if the mask a fulfils some algebraic properties, and the JSR of a certain set of matrices constructed from values in a is less than 1.

“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]



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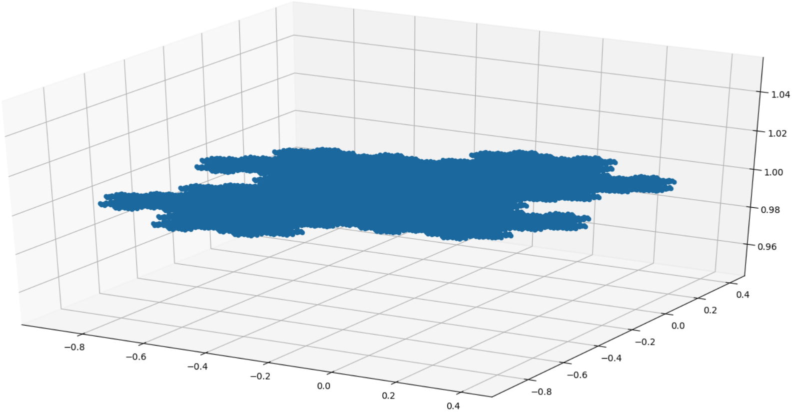

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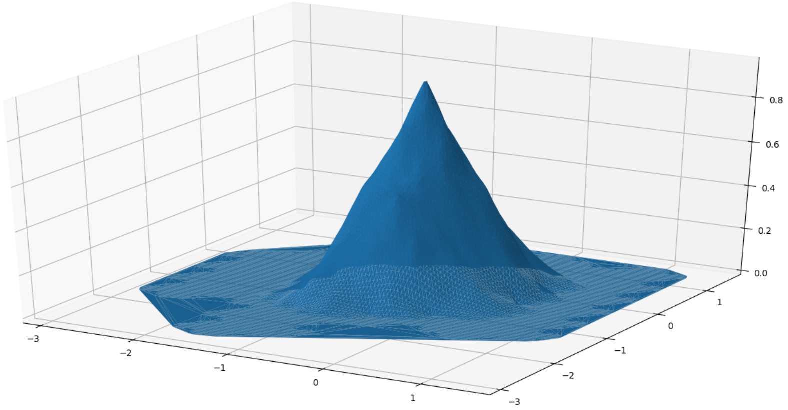

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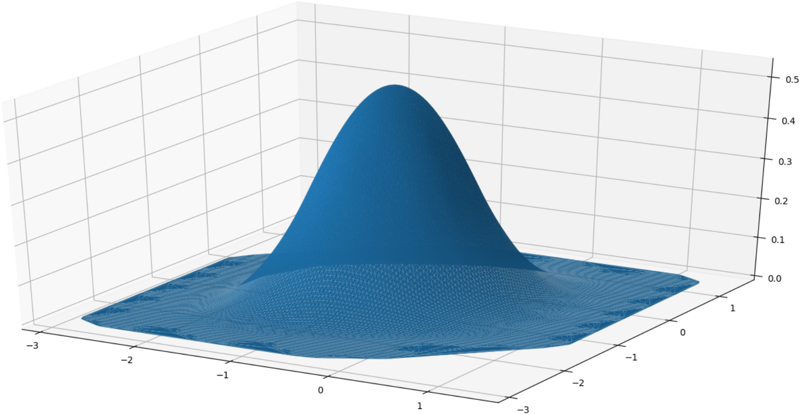

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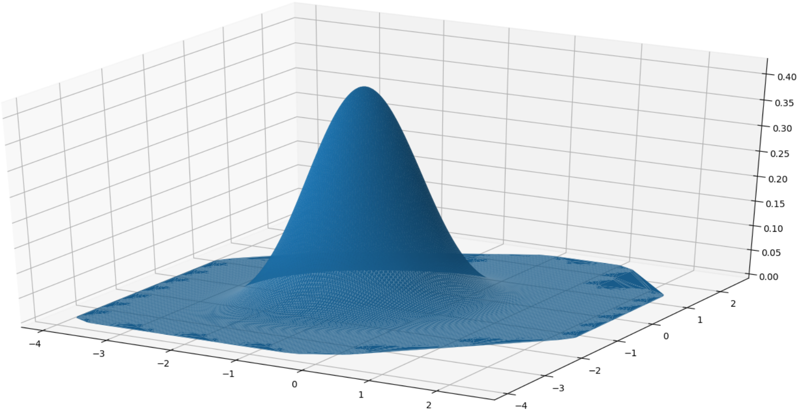

"Haar 2-tiles" in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]

Order	1	2	3	4
Square	0	1	2	3
Dragon				
Bear				

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Order	1	2	3	4
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Dragon	0	0.4764	1.558	2.192
Bear				

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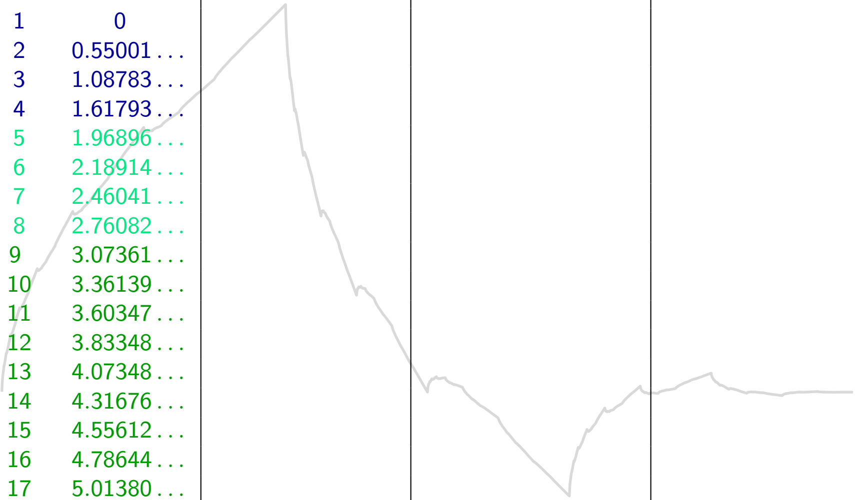
If you tame the dragon
you get a bear
which is smoother than the square

Kai Hörmann

Hölder regularity α of Daubechies wavelets



n	α
1	0
2	0.55001 . . .
3	1.08783 . . .
4	1.61793 . . .
5	1.96896 . . .
6	2.18914 . . .
7	2.46041 . . .
8	2.76082 . . .
9	3.07361 . . .
10	3.36139 . . .
11	3.60347 . . .
12	3.83348 . . .
13	4.07348 . . .
14	4.31676 . . .
15	4.55612 . . .
16	4.78644 . . .
17	5.01380 . . .
18	5.23917 . . .
19	5.46532 . . .
20	5.69108 . . .

Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16];

Hölder regularity α of Daubechies wavelets

n	α	20	5.69108 . . .	40	10.07073 . . .
1	0	21	5.91500 . . .	41	10.28656 . . .
2	0.55001 . . .	22	6.13779 . . .	42	10.50220 . . .
3	1.08783 . . .	23	6.35958 . . .		
4	1.61793 . . .	24	6.58096 . . .		
5	1.96896 . . .	25	6.80198 . . .		
6	2.18914 . . .	26	7.02250 . . .		
7	2.46041 . . .	27	7.24241 . . .		
8	2.76082 . . .	28	7.46187 . . .		
9	3.07361 . . .	29	7.68091 . . .		
10	3.36139 . . .	30	7.89962 . . .		
11	3.60347 . . .	31	8.11801 . . .		
12	3.83348 . . .	32	8.33605 . . .		
13	4.07348 . . .	33	8.55379 . . .		
14	4.31676 . . .	34	8.77123 . . .		
15	4.55612 . . .	35	8.98841 . . .		
16	4.78644 . . .	36	9.20533 . . .		
17	5.01380 . . .	37	9.42202 . . .		
18	5.23917 . . .	38	9.63847 . . .		
19	5.46532 . . .	39	9.85474 . . .		

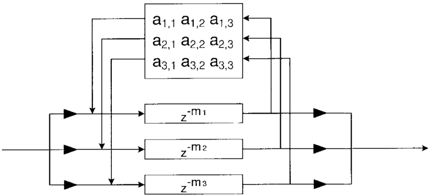
Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16]; M.;

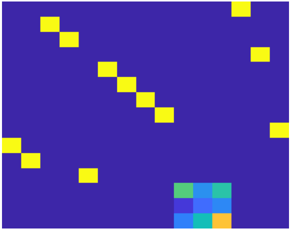
Hölder regularity α of Daubechies wavelets

n	α							
		20	5.69108 . . .	40	10.07073 . . .	60	≤ 14.35900 . . .	8
1	0	21	5.91500 . . .	41	10.28656 . . .	61	≤ 14.57219 . . .	8
2	0.55001 . . .	22	6.13779 . . .	42	10.50220 . . .	62	≤ 14.78531 . . .	8
3	1.08783 . . .	23	6.35958 . . .	43	≤ 10.71766 . . .	63	≤ 14.99833 . . .	8
4	1.61793 . . .	24	6.58096 . . .	44	≤ 10.93295 . . .	64	≤ 15.21127 . . .	8
5	1.96896 . . .	25	6.80198 . . .	45	≤ 11.14807 . . .	65	≤ 15.42413 . . .	8
6	2.18914 . . .	26	7.02250 . . .	46	≤ 11.36304 . . .	66	≤ 15.63692 . . .	8
7	2.46041 . . .	27	7.24241 . . .	47	≤ 11.57785 . . .	67	≤ 15.84962 . . .	8
8	2.76082 . . .	28	7.46187 . . .	48	≤ 11.79252 . . .	68	≤ 16.06226 . . .	8
9	3.07361 . . .	29	7.68091 . . .	49	≤ 12.00705 . . .	69	≤ 16.27482 . . .	8
10	3.36139 . . .	30	7.89962 . . .	40	≤ 12.22144 . . .	70	≤ 16.48731 . . .	9
11	3.60347 . . .	31	8.11801 . . .	51	≤ 12.43571 . . .	71	≤ 16.69973 . . .	9
12	3.83348 . . .	32	8.33605 . . .	52	≤ 12.64985 . . .	72	≤ 16.91209 . . .	9
13	4.07348 . . .	33	8.55379 . . .	53	≤ 12.86387 . . .	73	≤ 17.12438 . . .	9
14	4.31676 . . .	34	8.77123 . . .	54	≤ 13.07778 . . .	74	≤ 17.33661 . . .	9
15	4.55612 . . .	35	8.98841 . . .	55	≤ 13.29157 . . .	75	≤ 17.54878 . . .	9
16	4.78644 . . .	36	9.20533 . . .	56	≤ 13.50526 . . .	76	≤ 17.76089 . . .	9
17	5.01380 . . .	37	9.42202 . . .	57	≤ 13.71884 . . .	77	≤ 17.97295 . . .	9
18	5.23917 . . .	38	9.63847 . . .	58	≤ 13.93232 . . .	78	≤ 18.18494 . . .	9
19	5.46532 . . .	39	9.85474 . . .	59	≤ 14.14571 . . .	79	≤ 18.39688 . . .	9

Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16]; M.; M.;

🍺: Audio Filters





ipa: Non-negative matrices [GP13]

Theorem (Perron, Frobenius)

If $A \in \mathbb{R}^s_{\geq 0}$, *then*

- $\rho(A)$ *is an eigenvalue*
- *Leading eigenvector is in* $\mathbb{R}^s_{\geq 0}$

Invariant cone is the key.

🍺: Non-negative matrices

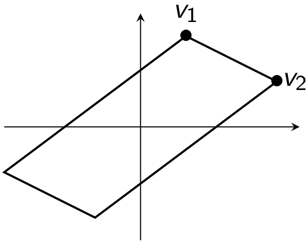
Definition

$$co_{-}(V) = \left\{x \in \mathbb{R}^s_+ : x = y - z, y \in \mathrm{co}(V), z \in \mathbb{R}^s_+\right\}$$

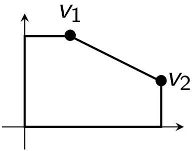
🍺: Non-negative matrices

Definition

$$co_-(V) = \left\{x \in \mathbb{R}^s_+ : x = y - z, y \in \mathrm{co}(V), z \in \mathbb{R}^s_+\right\}$$



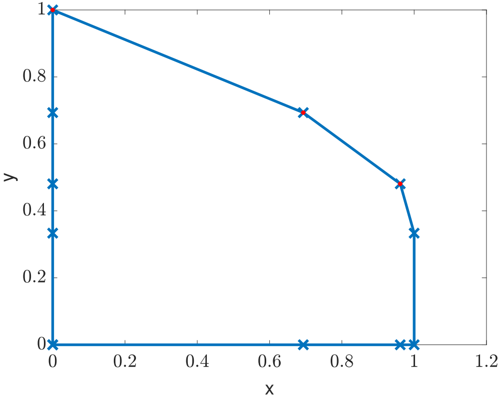
$\mathrm{co}_s(\{v_1, v_2\})$



$\mathrm{co}_-(\{v_1, v_2\})$

🍺: Non-negative matrices: Examples

```
T = {[1 1;0 1], [0 0;1 1]};
ipa( T, 'plot','polytope' )
```



Capacity of codes avoiding forbidden differences D

D	cap(D)
$\{\circ \pm \pm\}$	?
$\{\circ + - +\}$	0.879?
$\{\circ + + +\}$	0.879?
$\{\circ + \circ \pm\}$	?
$\{\circ + - + \circ\}$	0.9162?
$\{\circ + + + \circ\}$	0.9162?
$\{\circ + + + + \circ\}$	?
$\{+ + + + + - \circ\}$	?

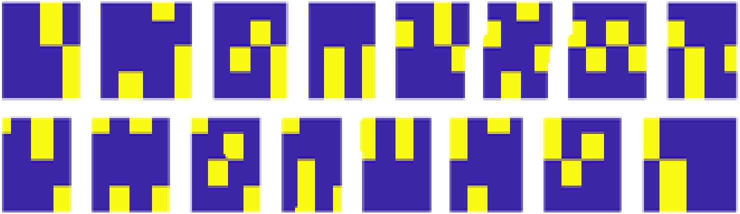
- D: Set of restrictions
- cap(D): relative number of binary words under the restrictions

Capacity of codes avoiding forbidden differences D

D	$\mathrm{cap}(D)$
$\{\circ \pm \pm\}$	$1/2$
$\{\circ + - +\}$	$0.879146421606638\ldots$
$\{\circ + + +\}$	$0.879146421606638\ldots$
$\{\circ + \circ \pm\}$	$2/3$
$\{\circ + - + \circ\}$	$0.916253191790226\ldots$
$\{\circ + + + \circ\}$	$0.916253191790226\ldots$
$\{\circ + + + + \circ\}$	$0.961407451709134\ldots$
$\{+ + + + + - \circ\}$	$0.976120327934917\ldots$

- D: Set of restrictions
- $\mathrm{cap}(D)$: relative number of binary words under the restrictions

```
T = codecapacity( {[1 1 0],[1 -1 0]} )
ipa( T )
```



ipa: Complex leading eigenvalue [GWZ05, GP13, MP21]

Elliptic 🍺

Theorem (Elliptic 🍺: Guglielmi, M., Protasov)

Given a finite set of real square matrices $\mathcal{T} = \{T_1, \ldots, T_J\}$. If

- *there exist finitely many s.m.p.s $\Pi_1, \ldots, \Pi_N$,*
- *whose leading eigenvectors are unique and simple, up to the complex conjugate,*
- *and there exists a spectral gap at 1,*

then the Elliptic 🍺 terminates, and thus, computes the JSR exactly.

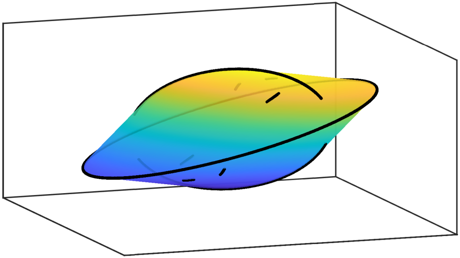
Definition of an ellipse using complex vectors

Definition

Given $a, b \in \mathbb{R}^s$, $v = a + ib$, we define

$$E(a, b) = E(v) = \{a \cos t + b \sin t : t \in [0, 2\pi]\} \subseteq \mathbb{R}^s.$$

An *elliptic polytope* is the convex hull of finitely many ellipses.



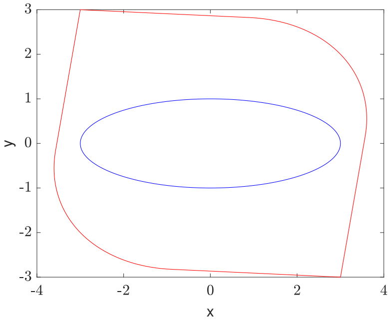
UniForm Object of ellipses

Applications 🍺: Examples

- None (for now)

Elliptic 🍺: ttoolboxes

```
V    = [3-2i 2+2i; -3 3].';
pt   = [3 1i].';
plotm( V, 'r-', 'hull',1, 'funct','c' )
plotm( pt, 'b-', 'funct','c', 'hold','on' )
polytopenorm( pt, V, 'c' )  % [0.84874  0.84874]
```



Lower spectral radius

🍺: Lower spectral radius:

- $\mathrm{LSR}(\mathcal{A}) = \lim_{n\to\infty} \inf_{A_j\in\mathcal{A}} \rho(A_n \cdots A_1)^{1/n}$
- LSR is not continuous

$$\mathcal{A}_k = \left\{ A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_k = \begin{bmatrix} 0 & 0 \\ -1/k & 1 \end{bmatrix} \right\}$$

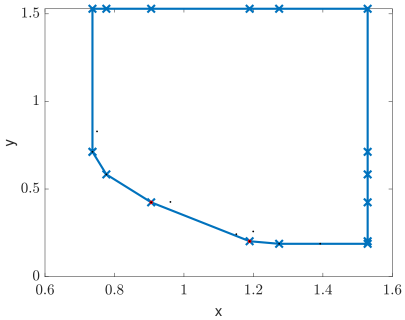
$$(A_0^k A_k)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathrm{LSR}(\{A_0, A_k\}) = 0$$

$$A_0^n A_\infty = \begin{bmatrix} 0 & n \\ 0 & 1 \end{bmatrix} \Rightarrow LSR(\{A_0, A_\infty\}) = 1$$

🍺: Lower spectral radius:

- LSR is continuous if the matrices share an invariant cone [GP13]

```
A = {[12 18;3 1], [11 5;3 15]};
ipa( A, 'case','lsr', 'plot','polytopeallpoint' )
```



Multi space ipa

Multi space 🍺: Motivation

Example

$$\begin{cases} x(k+1) = A(k)x(k) \\ x(0) = x_0 \end{cases}, A(k) \in \left\{ A_1 = \frac{1}{3},\ A_2 = 2 \right\}$$

$\text{JSR}(\{\frac{1}{3},\ 2\}) = 2 \Rightarrow$ unstable

Multi space 🍺: Motivation

Example

$$\begin{cases} x(k+1) = A(k)x(k) \\ x(0) = x_0 \end{cases} \quad , A(k) \in \left\{ A_1 = \frac{1}{3},\ A_2 = 2 \right\}$$

$\mathrm{JSR}(\{\frac{1}{3},\ 2\}) = 2 \Rightarrow$ unstable

Example

If $A_2 \cdot A_2$ is "forbidden", then "JSR"$(\{A_1, A_2\}) = \sqrt{\frac{2}{3}} \Rightarrow$ stable,

but $\nexists \|\cdot\| : \|A_j\| \leq \sqrt{\frac{2}{3}}$

Multi space 🍺: Construction [...]

(Multi) graph $\mathcal{G}$

- Nodes are lin. spaces L_i with a norm $||\cdot||_i$
- Edges are lin. op. between them
- paths in $\mathcal{G}$ are allowed matrix products
- multi norm $A_{ji} : L_i \to L_j$, $||A_{ji}|| = \sup_{||x||_i=1} ||A_{ji}x||_j$

Multi space 🍺: Construction [...]

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Example (Continued)

$\mathcal{G} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\mathcal{A} = \{1/3,\ 1/3,\ 2\}$

$||x||_{L_1} = |x|$, $||x||_{L_2} = \left|\frac{\lambda x}{2}\right|$, $\lambda = \mathrm{JSR}(\mathcal{A}, \mathcal{G}) = \sqrt{\frac{2}{3}}$

$||A_1||_{L_1 \to L_1} = \lambda$, $||A_2||_{L_1 \to L_2} = \lambda$, $||A_1||_{L_2 \to L_1} = \lambda$

```
ipa( {1/3 1/3 2}, [1 1 0;0 0 1;1 1 0] )
```

Multi space 🍺

[Cicone, Guglielmi, Protasov - 2018]

- A system corr. to $\mathcal{G}$ is stable if each trajectory goes to zero.
- $\mathrm{JSR}(\mathcal{A},\mathcal{G}) := \lim_{k\to\infty} \max \left|\left| A_{j_{k+1},j_k} \cdots A_{j_2,j_1} \right|\right|^{1/k}$
- $= \inf_{||\,.\,||} \{\lambda > 0 : ||A_{j,i}|| \leq \lambda\}$
- $\mathrm{JSR}(\mathcal{A},\mathcal{G}) < 1 \Leftrightarrow$ system is stable

Multi space 🍺: ODE

Theorem (Kamalov, M., Protasov)

• $(S) = \begin{cases} \dot{x}(t) = A(t)x(t) \\ x(0) = x_0 \end{cases}$

• *$A(t)$ is piecewise constant for at least for length m, and most for length M.*

• *$A(t) \in \mathcal{A}$ finite*

• *(S) is stable if* $\sigma_h - \frac{1}{m} \log \left(1 - \frac{\left|\left|(\mathcal{A}-\sigma_h I)^2\right|\right|}{8} h^2 \right) < 0$

$\sigma_h = \log \mathrm{JSR}(\mathcal{A}_h, \mathcal{G})$, $\mathcal{A}_h = \left\{ e^{mA_j}, e^{hA_j} : A_j \in \mathcal{A} \right\}$

Theorem

$\dot{x} = Ax$. *Then* $||x(\tau)|| \leq (1 - \frac{h^2}{8} \left|\left|A^2\right|\right|) \max \{||x(0)||, ||x(h)||\}$

Finite expressible tree algorithm (aka tree algorithm)

Computation: 🧀

- How does it work? Given: $\mathcal{A}$, Π, $||\cdot||$.

Computation: 🧀

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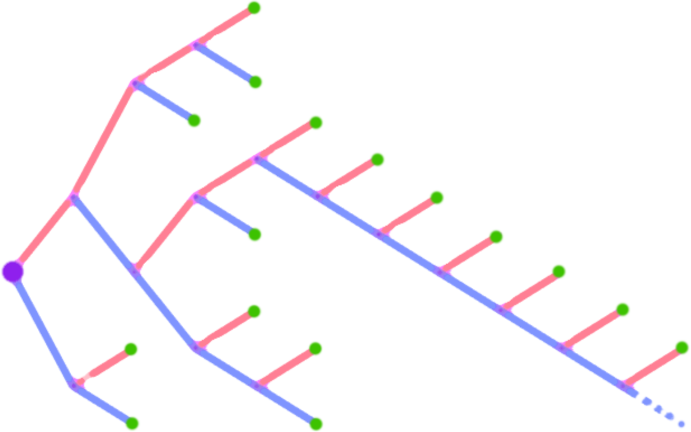
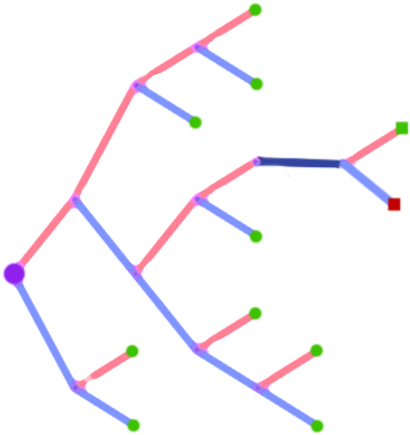


Computation: 🧀

- How does it work? Given: $\mathcal{A}$, Π, $||\cdot||$.



Computation:

Definition

Leafage $\mathcal{L}$ are all matrices in the green dots.

Theorem

If $||L|| \leq 1$ *for all* $L \in \mathcal{L}$, *then* JSR ≤ 1.

Theorem ([MR])

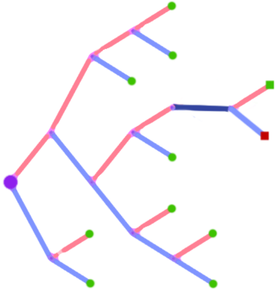
Given finite $\mathcal{A} \subseteq \mathbb{R}^{s \times s}$, *with a spectral gap and one s.m.p., then there exists a finite expressible tree.*

Ipa flavoured feta
Feta flavoured ipa

- $V \subseteq W \Rightarrow \|\cdot\|_W \leq \|\cdot\|_V$

Theorem (M., Reif)

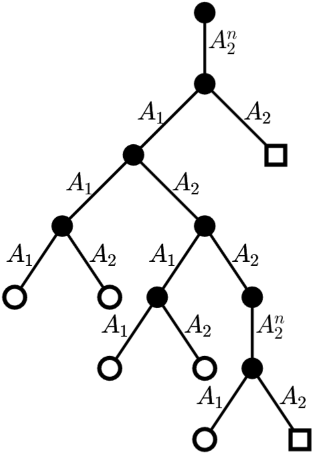
If there exists a finite set of vectors $V \subset \mathbb{R}^s$ *spanning* $\mathbb{R}^s$ *and such that* $LV \subset \mathrm{co}_s(V)$ *for all* $L \in \mathcal{L}(\mathbf{T})$ *then* $\mathrm{JSR}(\mathcal{A}) \leq 1$.



Example

$$A_1 = \frac{1}{\sqrt{13}} \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$A_2 = \frac{1}{\sqrt{13}} \begin{bmatrix} 1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$



Application?: Common invariant kone

Common invariant kone

Definition

$K \subseteq \mathbb{R}^s$ is a proper kone iff

- $K + K \subseteq K$
- $\mathbb{R}^+ K \subseteq K$
- K is closed
- $K \cap -K = \{0\}$
- $K^\circ \neq \emptyset$

Definition

K is common invariant kone for $\mathcal{A}$ iff $AK \subseteq K$ for all $A \in \mathcal{A}$

Common invariant kone

Definition

For $v \in \mathbb{R}^s$ and $K \subseteq \mathbb{R}^s$, we define the kone-(vector-)norm $||\cdot||_{K,v} = ||\cdot||_K$ via

$$||p||_K = \inf \{t > 0 : x \in E_{t,v} K\} \tag{1}$$

where

$$E_{t,v} = \begin{bmatrix} v & v^\perp \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & t & & \\ & & \ddots & \\ 0 & & & t \end{bmatrix} \begin{bmatrix} v & v^\perp \end{bmatrix}^T \in \mathbb{R}^{s \times s}, \tag{2}$$

Kone norm

Lemma

The norm $||\cdot|| = ||\cdot||_{K,v}$ *fulfils the following properties for all* $(x, v) > 0$

1. $||x||_K = 0 \Leftrightarrow x = \lambda v$ *for some* $\lambda \geq 0$
2. $||x||_K = ||\lambda x||_K = ||x||_{\lambda K}$ *for all* $\lambda > 0$
3. $t\,||x||_K = ||E_t x||_K$ *for all* $t > 0$
4. $t\,||x||_K = ||x||_{E_{t^{-1}} K}$ *for all* $t > 0$
5. $||x + y||_K \leq ||x||_K + ||y||_K$

Common invariant kone: Application

Definition

$$\mathrm{JSR}_1(\mathcal{A}) = \lim_{n\to\infty} \frac{1}{J}\left(\sum ||A_{j_k}\cdots A_{j_1}||\right)^{1/k}$$

$$\bar{A} = \frac{1}{J}\sum A_j$$

Theorem (Protasov 2008,2010)

$\mathcal{A}$ has common invariant cone $\Leftrightarrow \mathrm{JSR}_1(\mathcal{A}) = \rho(\bar{A})$

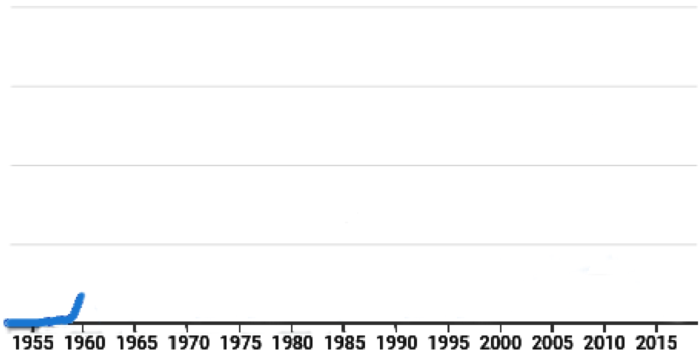
History

History

Gian-Carlo Rota, Gilbert Strang,
A note on the joint spectral radius, 1960

The notion of joint spectral radius of a set of elements of a normed algebra, introduced below, was obtained in the course of some work in matrix theory. It was later noticed that the same considerations are valid in any normed algebra, irrespective of dimension. The notion seems to be useful enough in certain contexts to warrant the following elementary discussion.

History



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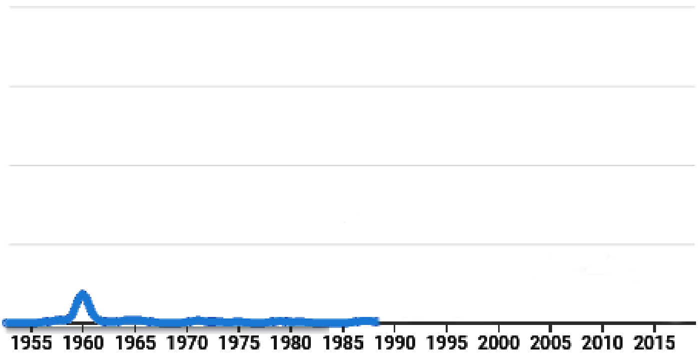
History



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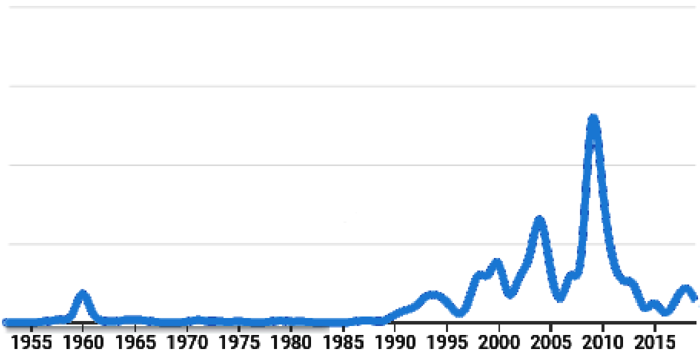
History

Ingrid Daubechies, Jeffrey C. Lagaris, *Sets of Matrices All Infinite Products of Which Converge*, 1992

An infinite product $\Pi_{i=1}^{\infty} M_i$ of matrices converges (on the right) if $\lim_{i\to\infty} M_1 \cdots M_i$ exists. [...] Such sets of matrices arise in constructing self-similar objects like von Koch's snowflake curve, in various interpolation schemes, in constructing wavelets of compact support, and in studying nonhomogeneous Markov chains. This paper gives necessary conditions and also some sufficient [...]

History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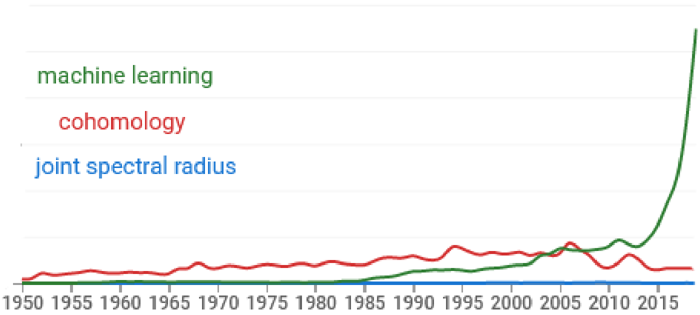

machine learning
cohomology
joint spectral radius
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Thank you for listening
Questions welcome