A JSR primer and the feta flavoured IPA

Höchst, 14^{th}-15^{th} March 2022

2022-03-14

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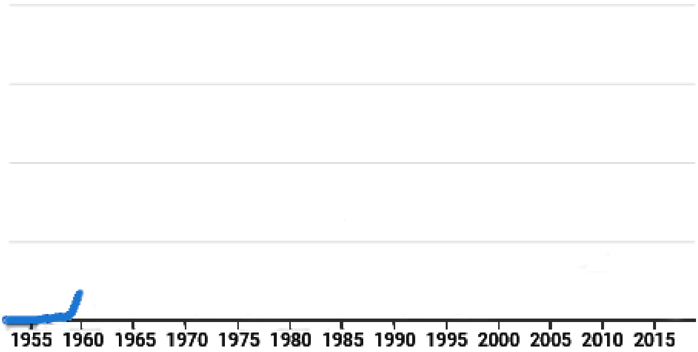
History

History

Gian-Carlo Rota, Gilbert Strang,
A note on the joint spectral radius, 1960

The notion of joint spectral radius of a set of elements of a normed algebra, introduced below, was obtained in the course of some work in matrix theory. It was later noticed that the same considerations are valid in any normed algebra, irrespective of dimension. The notion seems to be useful enough in certain contexts to warrant the following elementary discussion.

History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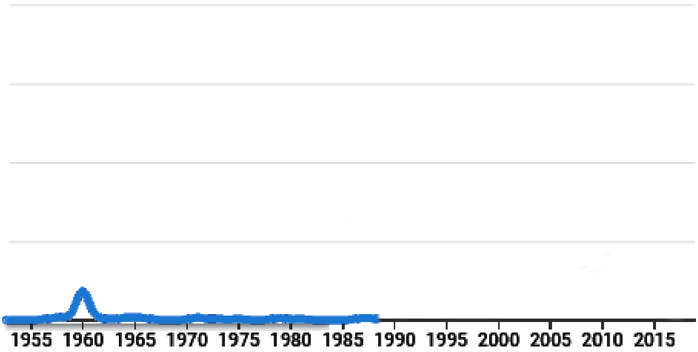
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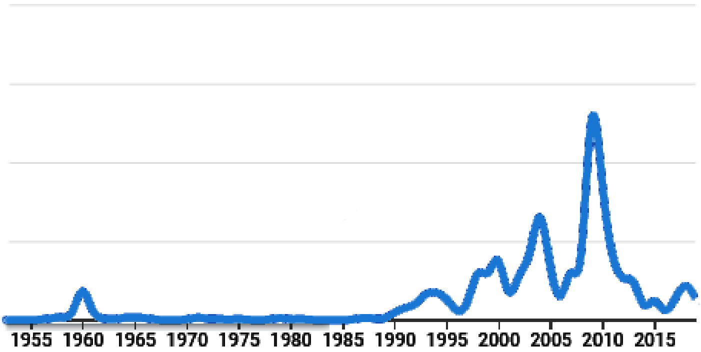
Ingrid Daubechies, Jeffrey C. Lagaris, *Sets of Matrices All Infinite Products of Which Converge*, 1992

An infinite product $\Pi_{i=1}^{\infty} M_i$ of matrices converges (on the right) if $\lim_{i\to\infty} M_1 \cdots M_i$ exists. [...] Such sets of matrices arise in constructing self-similar objects like von Koch's snowflake curve, in various interpolation schemes, in constructing wavelets of compact support, and in studying nonhomogeneous Markov chains. This paper gives necessary conditions and also some sufficient [...]

History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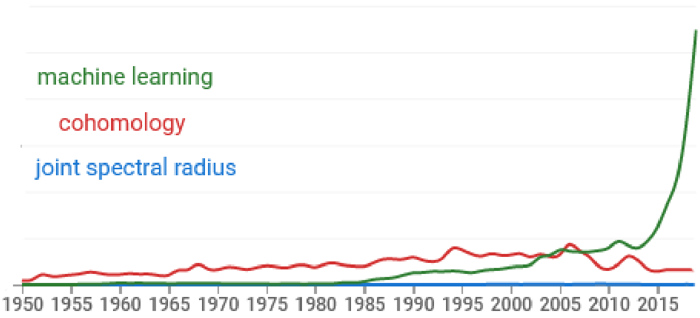
machine learning
cohomology
joint spectral radius
1950 1955 1960 1965 1970 1975 1980 1985 1990 1995 2000 2005 2010 2015

Basics

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Theorem (*Spectral radius*, Israel Gelfand, 1941)

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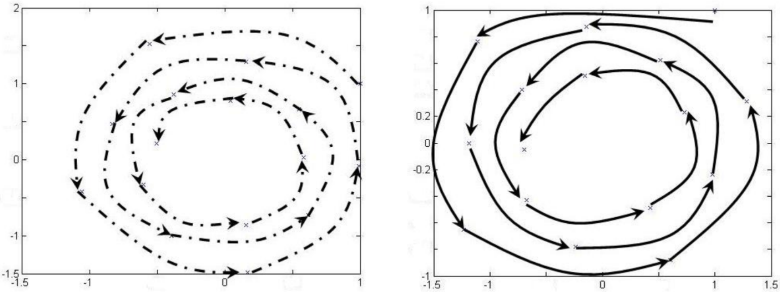
Definition (*Sub multiplicativity*)

$$||AB|| \leq ||A||\,||B||$$

Definition (*JSR*, Gian-Carlo Rota, Gilbert Strang, 1960)

$$\mathrm{JSR}(\mathcal{A}) := \lim_{n\to\infty} \max_{A_j\in\mathcal{A}} ||A_{j_n}\cdots A_{j_1}||^{1/n}, \quad \mathcal{A}\subseteq\mathbb{R}^{s\times s},\ \text{bounded}$$

Examples: JSR



Picture from: Raphaël M. Jungers, *The Joint Spectral Radius*, 2009

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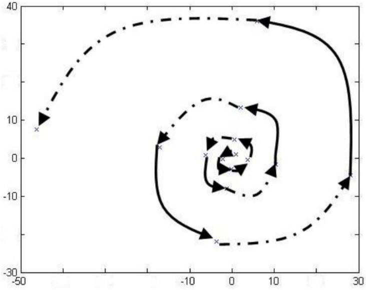

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$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{A} = \{A_1, A_2\}$$

$\mathrm{JSR}(\mathcal{A}) = 1$ $\qquad (A_1 A_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]).$

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$\rho(A_1) = \rho(A_2) = 0.$

Basic properties

Definition

$\mathrm{JSR}(\mathcal{A}) := \lim_{n\to\infty} \max_{A_j\in\mathcal{A}} ||A_{j_n} \cdots A_{j_1}||^{1/n}$

Theorem

- *Scaling:* $\mathrm{JSR}(\lambda\mathcal{A}) = \lambda\,\mathrm{JSR}(\mathcal{A})$, $\lambda \in \mathbb{R}$.

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Basic properties: Infimum of norms

Theorem

$\mathrm{JSR}(\mathcal{A}) = \inf_{||\cdot||} \sup_{A\in\mathcal{A}} ||A||$

Proof.

- Let $\varepsilon > 0$, $\tilde{\mathcal{A}} = (\mathrm{JSR}(\mathcal{A}) + \varepsilon)^{-1}\mathcal{A}$
- $||v||_\varepsilon = \max_{n\in\mathbb{N}_0} \left|\left|\tilde{A}_{j_n} \cdots \tilde{A}_{j_1} v\right|\right|_2$
- $\left|\left|\tilde{A}\right|\right|_\varepsilon \leq 1$ for all $\tilde{A} \in \tilde{\mathcal{A}}$
- $||A||_\varepsilon \leq \mathrm{JSR}(\mathcal{A}) + \varepsilon$

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Corollary

JSR *is continuous in* $\mathcal{A}$

Corollary

Numerical computation makes sense.

Basic property: Irreducibility

Definition (Irreducibility)

- If there exists $V \in \mathbb{R}^{s\times s}$ such that $VA_jV^{-1} = \begin{bmatrix} B_j & C_j \\ 0 & D_j \end{bmatrix}$ for all $A_j \in \mathcal{A}$, then $\mathcal{A}$ is reducible.

In other words: There exists no common invariant subspace (except $\{0\}$ and $\mathbb{R}^s$).

Theorem

If $\mathcal{A}$ is reducible, then $\mathrm{JSR}(\mathcal{A}) = \max\{\mathrm{JSR}(\mathcal{B}), \mathrm{JSR}(\mathcal{D})\}$.

Interesting properties

Interesting property: Generalized JSR

Theorem (The JSR theorem, [BW92])

If $\mathcal{A}$ is bounded, then

$$\mathrm{JSR}(\mathcal{A}) = \limsup_{n\to\infty} \rho(A_{j_n} \cdots A_{j_1})^{1/n}$$

Interesting property: Three member inequality

Theorem ([DL])

$$\max_{A_j \in \mathcal{A}} \rho\left(A_{j_k} \cdots A_{j_1}\right)^{1/k} \leq \mathrm{JSR}(\mathcal{A}) \leq \max_{A_j \in \mathcal{A}} \left|\left|A_{j_k} \cdots A_{j_1}\right|\right|^{1/k}.$$

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Corollary

$$\max_{A \in \mathcal{A}} \rho(A) \leq \mathrm{JSR}(\mathcal{A}) \leq \max_{A \in \mathcal{A}} ||A||.$$

Interesting property: Irreducibility

Theorem ([BW92])

Given a bounded set of matrices $\mathcal{A} \subseteq \mathbb{R}^{s\times s}$. For $v \in \mathbb{R}^s$ define

$$P(v) = \operatorname{co} \bigcup_{A_j \in \mathcal{A},\ n \in \mathbb{N}} \{\pm A_{j_n} \dots A_{j_1} v\}.$$

1. *If* $\mathrm{JSR}(\mathcal{A}) \geq 1$ *and for some* $v \in \mathbb{R}^s$ *the set* $P(v)$ *is bounded and has non-empty interior, then* $\mathcal{A}$ *is irreducible and* $\mathrm{JSR}(\mathcal{A}) = 1$.
2. *If* $\mathcal{A}$ *is irreducible and* $\mathrm{JSR}(\mathcal{A}) = 1$, *then* $P(v)$ *is a bounded subset of* $\mathbb{R}^s$ *for any* $v \in \mathbb{R}^s$.

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$\rightarrow$ $P(v)$ is the unit ball of a norm

Interesting property: Finiteness conj. [LW95]

Definition (s.m.p.)

- If there exists $\Pi = A_{j_N} \cdots A_{j_1}$ such that $\rho(\Pi)^{1/N} = \mathrm{JSR}(\mathcal{A})$, the set $\mathcal{A}$ possesses the *finiteness property*.
- Π is called an *spectral maximizing product (s.m.p.)*.

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Theorem (Finiteness property, [BM02, HST10])

- *There exist infinite many values $\alpha \in [0,1]$ for which $\{\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right], \alpha\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]\}$ does not posses the finiteness property.*
- *An explicit value is (¿$\alpha_0 \in \mathbb{R}$ or $\mathbb{Q}$?)*

$$\alpha_0 = 0.749326546330367557943961948091344672091327\ldots$$

Interesting property: Irreducibility + Finiteness pr.

Definition

For any s.m.p. Π:

- $$\Pi \simeq V \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \lambda_K & 0 \\ 0 & 0 & 0 & T \end{bmatrix} V^{-1}, \quad \rho(T) < |\lambda_k|\,, \ T\text{: triangular.}$$
- λ_k are *leading eigenvalues*
- Corresponding eigenvectors are *leading eigenvectors*

Interesting property: Irreducibility + Finiteness pr.

Theorem (GP2016)

Given irreducible $\mathcal{A} \subseteq \mathbb{R}^{s\times s}$, $\mathrm{JSR}(\mathcal{A}) = 1$, *s.m.p.* Π, *with leading right/left eigenvectors* $v \in \mathbb{R}^s$, $w \in \mathbb{R}^{1\times s}$,

$$\begin{matrix} \Pi v = \lambda v \\ w\Pi = \lambda w \end{matrix} \quad , \ |\lambda| = 1.$$

Then,[1]

- $(v, w^T) \neq 0$
- $P(v) \subseteq \left\{ x \in \mathbb{R}^s : (w^T, x) \leq (v, x) \right\}$

[1]This statement could be wrong if $v \in \mathbb{C}^s$.

Example: Unit ball of a norm

$$A_1 = \lambda^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \lambda^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad \lambda = 4\sqrt{3} + 7$$

$$\mathcal{A} = \{A_1, A_2\}$$

$$\Pi = A_2 A_1 A_2^2 A_1 A_2 A_1^2, \quad \rho(\Pi) = 1$$

$$v = \left[\sqrt{3} + 1 \quad 1\right]^T, \quad w = \left[(\sqrt{3} + 1)/2 \quad 1\right]$$

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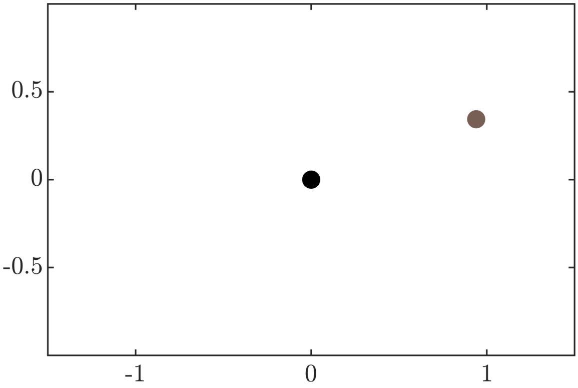

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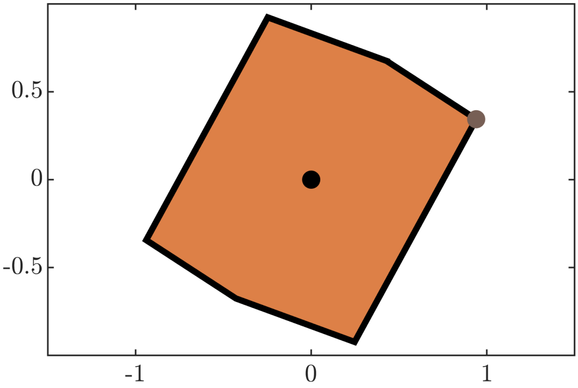

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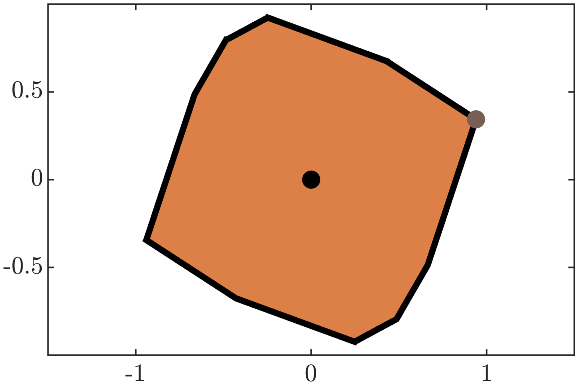

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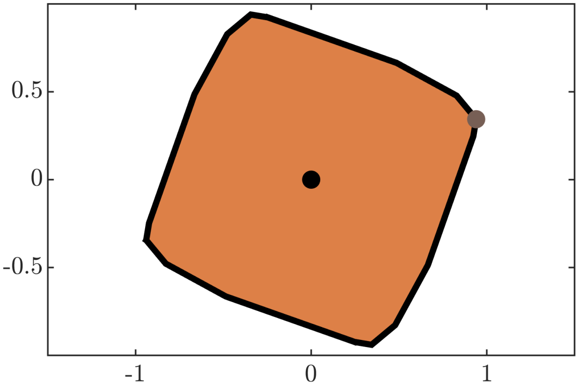

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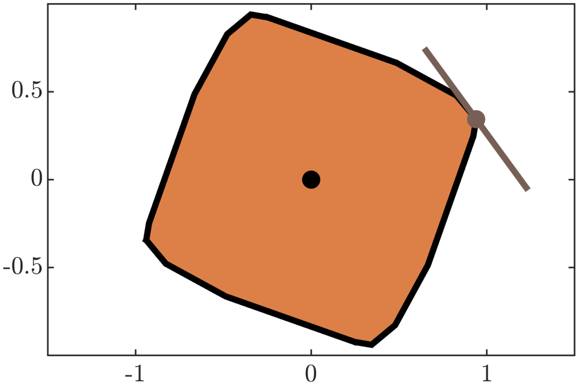

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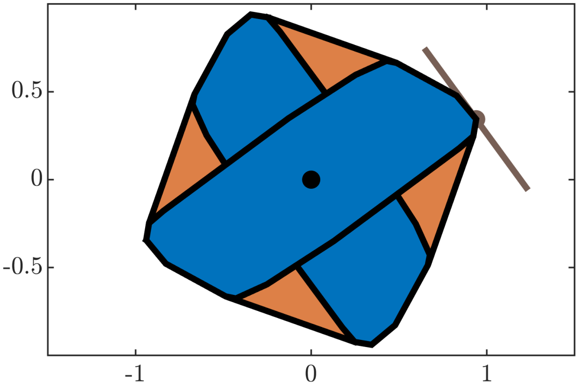

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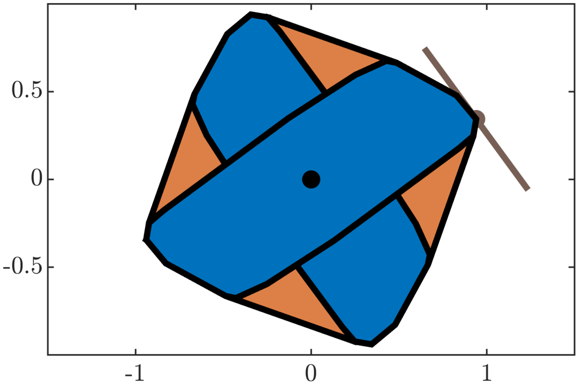

$||A||_{P(v)} = 1, ||B||_{P(v)} = 1.$

Interesting property: Spectral gap

Definition

Given bounded $\mathcal{A} \subseteq \mathbb{R}^{s \times s}$ with $\rho = \mathrm{JSR}(\mathcal{A}) > 0$, and let $\tilde{\mathcal{A}} = \{A_j/\rho : A_j \in \mathcal{A}\}$.
$\mathcal{A}$ has a spectral gap if for every product $\tilde{\Pi} = \tilde{A}_{j_n} \cdots \tilde{A}_{j_1}$, $\tilde{A}_j \in \tilde{\mathcal{A}}$, which is not an s.m.p., there exists $\gamma < 1$ such that $\rho(\tilde{\Pi}) < \gamma$.

Computing the JSR

Computing the JSR: Algorithms

- Lower bounds (*Modified Gripenberg [M]*, *Genetic* [Chang])
- Upper bounds
- Lower and upper bounds

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- Brute force ([Brutus 42BC])
- Divide et impera ([Philipp II, 336BC], *Gripenberg*, Mod. Grip., 🧀)
- Choose family of norms and search for the best (sos [PJ08], 🍺)
- Other stuff (pathcomplete [Parrilo, Roozbehani])

Computation: Modified Gripenberg algorithm

Gripenberg algorithm [Gripenberg, 1996], based on

$$\max_{(j)\in\{1,\dots,J\}^k} \rho(A_{j_k}\cdots A_{j_1})^{1/k} \leq \mathrm{JSR}(\mathcal{A}) \leq \max_{(j)\in\{1,\dots,J\}^k} ||A_{j_k}\cdots A_{j_1}||^{1/k}.$$

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• Gripenberg algorithm sorts out matrix products,
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• Number of matrix products to be computed are kept reasonably small.

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Given $\mathcal{A} = \{A_j \in \mathbb{R}^{s\times s} : j = 1,\ldots,J\}$.

• Modified Gripenberg algorithm sorts out matrix products, of which we hope that they have averaged spectral radius less than the joint spectral radius of the given set.

• Number of matrix products to be computed are kept bounded.

Examples: Modified Gripenberg algorithm

$\mathcal{A} = \{A_j \in \mathbb{R}^{8\times 8}$: random entries, $j = 1, \dots, 8\}$
100 test runs

Algorithm	time	success
mod. Gripenberg	5.4 s	100%
Gripenberg	82.1 s	100%
genetic	12.0 s	87%
brute force	180.0 s	74%

success: success rate of finding a product which attains the JSR.
time: runtime

Computation: 🍺

- How does it work?

Computation: 🍺

- How does it work?
- Why does it work?

Theorem (🍺 [GP])

Given finite $\mathcal{A} \subseteq \mathbb{R}^{s \times s}$, with a spectral gap, finitely many s.m.p.s, and each of which having a unique leading eigenvalue λ_i. Then there exist leading eigenvectors v_i such that

$$\operatorname{co} \bigcup P(v_i)$$

has finitely many vertices.

Computation:

- How does it work?

Computation:

- How does it work? Given:
  - Sheep milk
  - max. 30% Goat milk
  - Rennet (dt.: Lab)
  - Salt water, 7%

Computation: 🧀

- How does it work? Given: $\mathcal{A}$, $||\cdot||$, Π.

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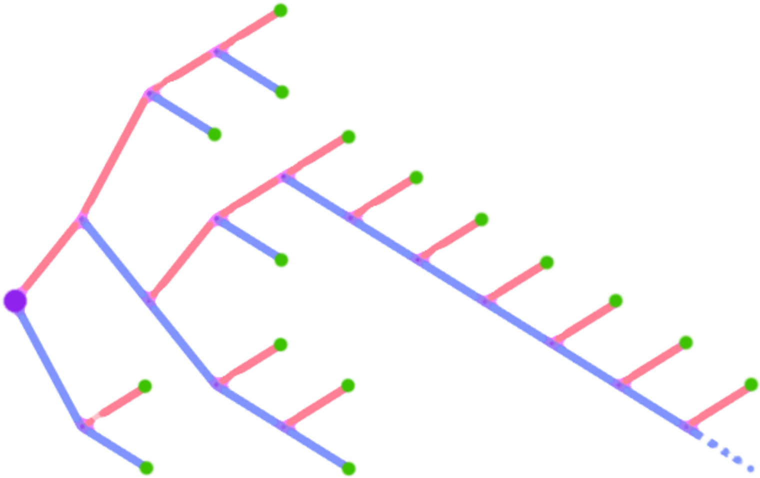

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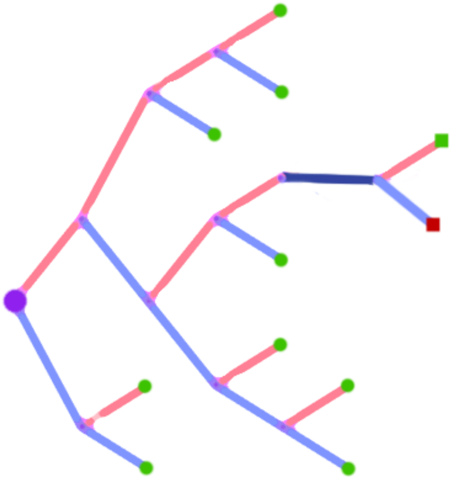

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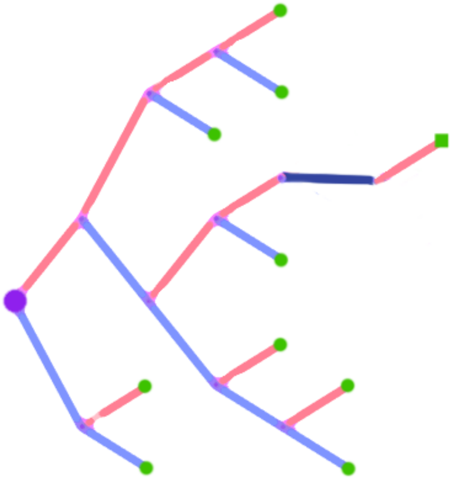


Computation:

- Why does it work?

Theorem ([MR])

Given finite $\mathcal{A} \subseteq \mathbb{R}^{s \times s}$, with a spectral gap and one s.m.p.s, then there exists a finite expressible tree.

Computation: 🧀 vs. 🍺

- Termination criteria:

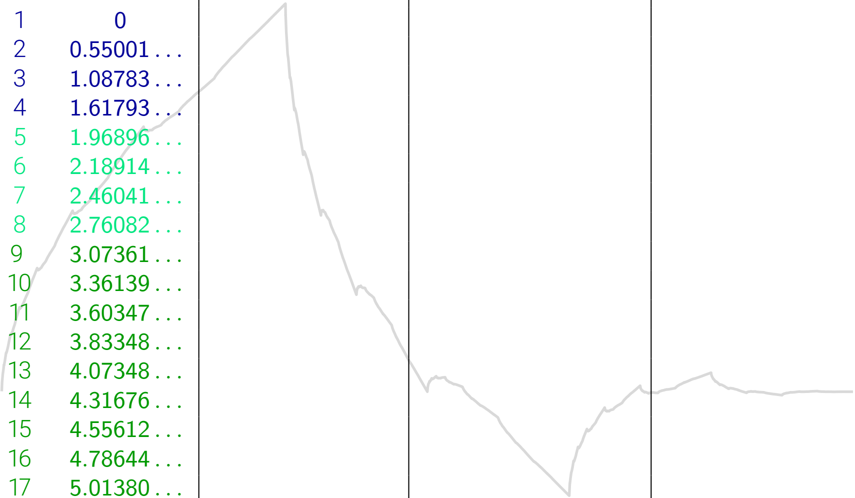
	🍺	🧀
# s.m.p.s	finitely many	1
# leading eigenvectors	1 / s.m.p.	arbitrary
spectral gap	necessary	necessary

- Current work: 🧀 flavoured 🍺

Examples from Applications

Hölder regularity α of Daubechies wavelets

n	α
1	0
2	0.55001 . . .
3	1.08783 . . .
4	1.61793 . . .
5	1.96896 . . .
6	2.18914 . . .
7	2.46041 . . .
8	2.76082 . . .
9	3.07361 . . .
10	3.36139 . . .
11	3.60347 . . .
12	3.83348 . . .
13	4.07348 . . .
14	4.31676 . . .
15	4.55612 . . .
16	4.78644 . . .
17	5.01380 . . .
18	5.23917 . . .
19	5.46532 . . .
20	5.69108 . . .



Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16];

Hölder regularity α of Daubechies wavelets

n	α	n	α	n	α
1	0	20	5.69108 ...	40	10.07073 ...
2	0.55001 ...	21	5.91500 ...	41	10.28656 ...
3	1.08783 ...	22	6.13779 ...	42	10.50220 ...
4	1.61793 ...	23	6.35958 ...		
5	1.96896 ...	24	6.58096 ...		
6	2.18914 ...	25	6.80198 ...		
7	2.46041 ...	26	7.02250 ...		
8	2.76082 ...	27	7.24241 ...		
9	3.07361 ...	28	7.46187 ...		
10	3.36139 ...	29	7.68091 ...		
11	3.60347 ...	30	7.89962 ...		
12	3.83348 ...	31	8.11801 ...		
13	4.07348 ...	32	8.33605 ...		
14	4.31676 ...	33	8.55379 ...		
15	4.55612 ...	34	8.77123 ...		
16	4.78644 ...	35	8.98841 ...		
17	5.01380 ...	36	9.20533 ...		
18	5.23917 ...	37	9.42202 ...		
19	5.46532 ...	38	9.63847 ...		
		39	9.85474 ...		

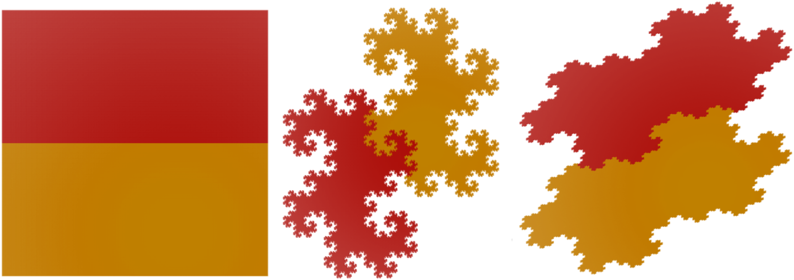
Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16]; M.;

Hölder regularity α of Daubechies wavelets

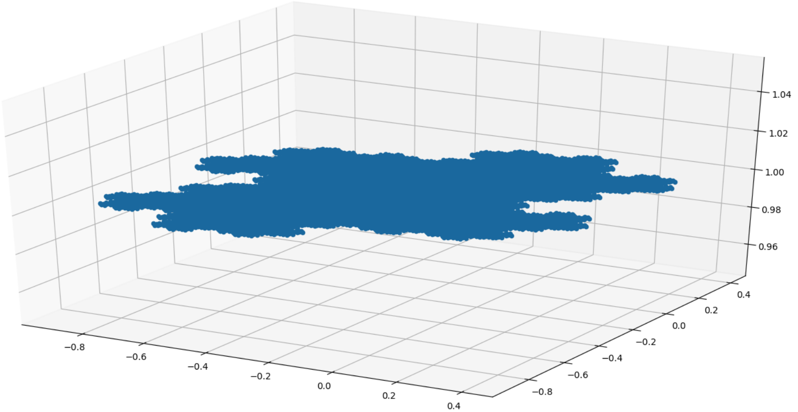
n	α	20	$5.69108\ldots$	40	$10.07073\ldots$	60	$\le 14.35900\ldots$	8
1	0	21	$5.91500\ldots$	41	$10.28656\ldots$	61	$\le 14.57219\ldots$	8
2	$0.55001\ldots$	22	$6.13779\ldots$	42	$10.50220\ldots$	62	$\le 14.78531\ldots$	8
3	$1.08783\ldots$	23	$6.35958\ldots$	43	$\le 10.71766\ldots$	63	$\le 14.99833\ldots$	8
4	$1.61793\ldots$	24	$6.58096\ldots$	44	$\le 10.93295\ldots$	64	$\le 15.21127\ldots$	8
5	$1.96896\ldots$	25	$6.80198\ldots$	45	$\le 11.14807\ldots$	65	$\le 15.42413\ldots$	8
6	$2.18914\ldots$	26	$7.02250\ldots$	46	$\le 11.36304\ldots$	66	$\le 15.63692\ldots$	8
7	$2.46041\ldots$	27	$7.24241\ldots$	47	$\le 11.57785\ldots$	67	$\le 15.84962\ldots$	8
8	$2.76082\ldots$	28	$7.46187\ldots$	48	$\le 11.79252\ldots$	68	$\le 16.06226\ldots$	8
9	$3.07361\ldots$	29	$7.68091\ldots$	49	$\le 12.00705\ldots$	69	$\le 16.27482\ldots$	8
10	$3.36139\ldots$	30	$7.89962\ldots$	40	$\le 12.22144\ldots$	70	$\le 16.48731\ldots$	9
11	$3.60347\ldots$	31	$8.11801\ldots$	51	$\le 12.43571\ldots$	71	$\le 16.69973\ldots$	9
12	$3.83348\ldots$	32	$8.33605\ldots$	52	$\le 12.64985\ldots$	72	$\le 16.91209\ldots$	9
13	$4.07348\ldots$	33	$8.55379\ldots$	53	$\le 12.86387\ldots$	73	$\le 17.12438\ldots$	9
14	$4.31676\ldots$	34	$8.77123\ldots$	54	$\le 13.07778\ldots$	74	$\le 17.33661\ldots$	9
15	$4.55612\ldots$	35	$8.98841\ldots$	55	$\le 13.29157\ldots$	75	$\le 17.54878\ldots$	9
16	$4.78644\ldots$	36	$9.20533\ldots$	56	$\le 13.50526\ldots$	76	$\le 17.76089\ldots$	9
17	$5.01380\ldots$	37	$9.42202\ldots$	57	$\le 13.71884\ldots$	77	$\le 17.97295\ldots$	9
18	$5.23917\ldots$	38	$9.63847\ldots$	58	$\le 13.93232\ldots$	78	$\le 18.18494\ldots$	9
19	$5.46532\ldots$	39	$9.85474\ldots$	59	$\le 14.14571\ldots$	79	$\le 18.39688\ldots$	9

Daubechies, Lagarias, [92]; Gripenberg, [96]; Guglielmi, Protasov, [16]; M.; M.;

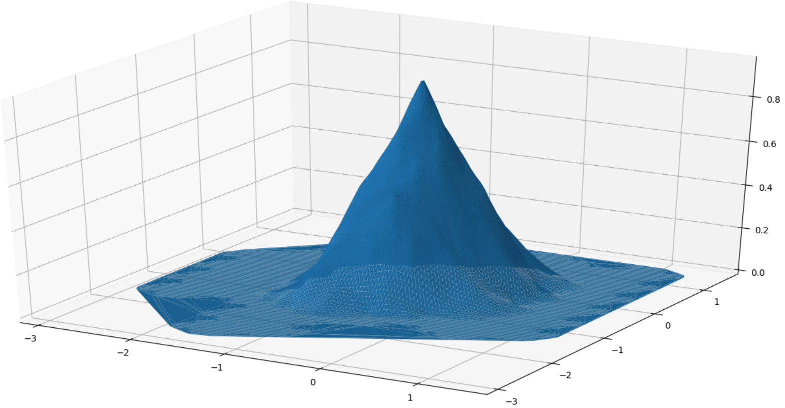
“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]



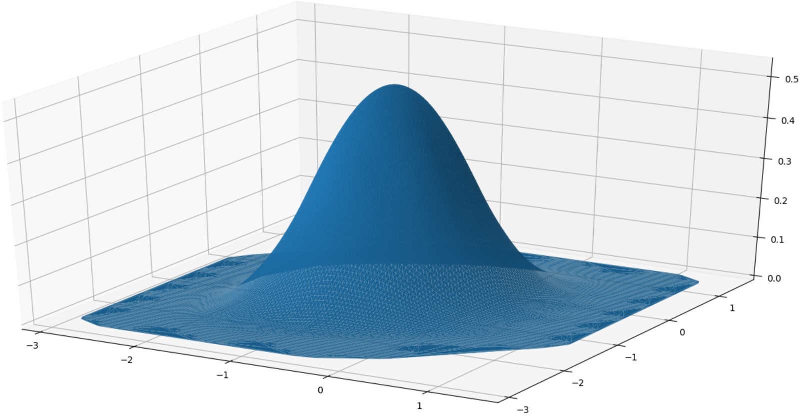
“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]



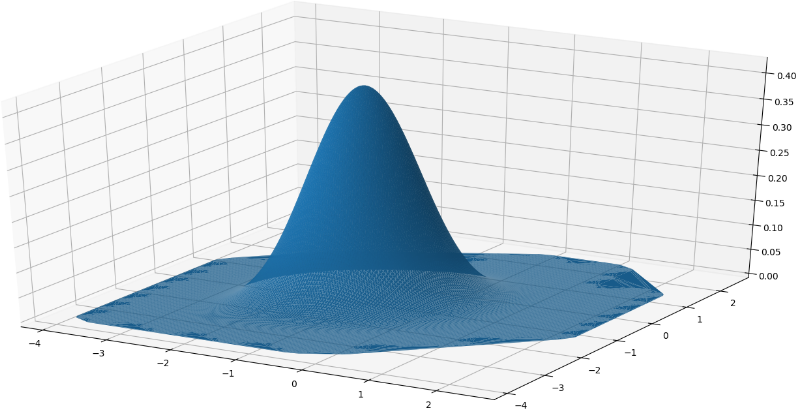
"Haar 2-tiles" in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]



"Haar 2-tiles" in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]



“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]



“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]

Order	1	2	3	4
Square	0	1	2	3
Dragon				
Bear				

“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]

Order	1	2	3	4
Square	0	1	2	3
Dragon	0	0.4764	1.558	2.192
Bear				

“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]

Order	1	2	3	4
Square	0	1	2	3
Dragon	0	0.4764	1.558	2.192
Bear	0			

“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]

Order	1	2	3	4
Square	0	1	2	3
Dragon	0	0.4764	1.558	2.192
Bear	0	0.7892		

“Haar 2-tiles” in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]

Order	1	2	3	4
Square	0	1	2	3
Dragon	0	0.4764	1.558	2.192
Bear	0	0.7892	2.235	

"Haar 2-tiles" in $\mathbb{R}^2$, Hölder regularity [T. Zaitseva]

Order	1	2	3	4
Square	0	1	2	3
Dragon	0	0.4764	1.558	2.192
Bear	0	0.7892	2.235	3.074

Finiteness conjecture

Theorem ([M22])

The finiteness conjecture holds for all pairs of matrices

(a) *of dimension 2 with entries* $\{0, 1\}$,

(b) *of dimension 2 with entries* $\{-1, 0, 1\}$, *and*

(c) *of dimension 3 with entries* $\{0, 1\}$.

ttoolboxes

- Matlab toolbox for computation of the joint spectral radius
- gitlab.com/tommsch/ttoolboxes
- mathematically rigorous results

```
T = {[1 1;2 1], [1 0;-1 1]};
tjsr( T );
```

Open research questions

Open questions

- Finiteness conjecture for rational matrices
- Does exist a set $\mathcal{A}$, coming from applications, with a complex leading eigenvector
- Do exist products which cannot be an s.m.p..
- Algorithm for sets with multiple s.m.p.s and multiple leading eigenvalues.
- Algorithm for sets without spectral gap
- Characterizations for the spectral gap
- Lower spectral radius

Thank you for listening
Questions welcome