Elliptic polytopes and invariant norms of linear operators

2021-11-11

T. Mejstrik / V. Yu. Protasov

Motivation: Subdivision schemes

Definition (Subdivision operator / scheme)

- Given *subdivision operator* $S = (a, M)$, $\rho(M^{-1}) < 1$, we define $S : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s)$,

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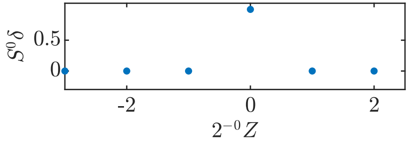

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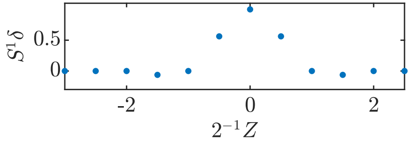

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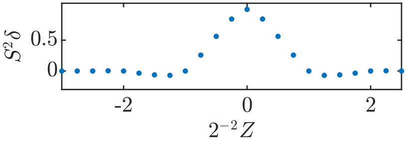

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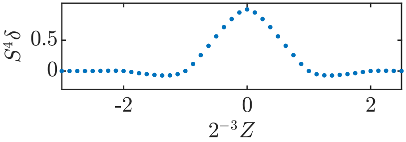

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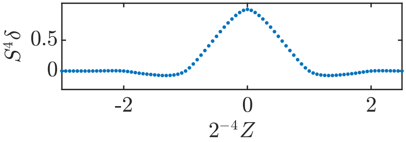

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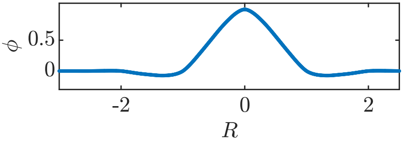

Convergence of subdivision schemes

Definition

A subdivision scheme S is *convergent* if $\exists \phi \in C_c^0(\mathbb{R}^s)$ such that

$$\lim_{n\to\infty} S^n\delta = \phi \Leftrightarrow \sup_{\alpha\in\mathbb{Z}^s} \left|\phi(M^{-r}\alpha) - S^r\delta(\alpha)\right| \leq C\rho^r.$$

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A subdivision scheme S is convergent $\Leftrightarrow \text{JSR}\left(\left\{B_1, \ldots, B_{|\det M|}\right\}\right) < 1$, *where B_j are finite square matrices constructed using $S = (a, M)$.*

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Definition (JSR)

$$\mathrm{JSR}\left(\{B_1, \ldots, B_J\}\right) := \lim_{n\to\infty} \sup_{B_j} ||B_{j_n} \cdots B_{j_1}||^{1/n}$$

Example of a subdivision scheme

Example (s.s. $S = (a, M)$, $a = \frac{1}{6}[-1 \quad 2 \quad 4 \quad 2 \quad 3 \quad 2]$, $M = 2$)

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$$A_0 = \frac{1}{6}\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 4 & 2 & -1 & 0 & 0 \\ 3 & 2 & 4 & 2 & -1 \\ 0 & 2 & 3 & 2 & 4 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}, A_1 = \frac{1}{6}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 2 & 4 & 2 & -1 & 0 \\ 2 & 3 & 2 & 4 & 2 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$V = [1 \quad 1 \quad 1 \quad 1 \quad 1]^{\perp}$$

$$B_0 = A_0|_V = \frac{1}{6}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}, B_1 = T_1|_V = \frac{1}{6}\begin{bmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 3 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\mathrm{JSR}(\{B_0, B_1\}) = \rho(B_0) = \rho(B_1) = 0.539338421077\ldots < 1$$

Exact computation of the JSR

- **I**nvariant **p**olytope **a**lgorithm [Guglielmi, M., Protasov, Wirth, Zennaro, . . .] (🍺)
- Infinite **tree** algorithm [Möller, Reif] (🌳)

🍺

Theorem (🍺)

Given a finite set of real square matrices $\mathcal{T} = \{T_1, \dots, T_J\}$. *If*

- *there exist finitely many s.m.p.s* $\Pi_1, \dots, \Pi_N$,
  i.e. $\rho(\Pi_n)^{1/len(\Pi_n)} = \mathrm{JSR}(\mathcal{T})$, $n = 1, \dots, N$,
- *whose leading eigenvalues are unique and simple,*
  i.e. $\Lambda_n^{-1}\Pi_n\Lambda_n = \begin{bmatrix} 1 & O \\ O & * \end{bmatrix}$, $n = 1, \dots, N$,
- *and there exists a spectral gap at* 1,
  i.e. $\exists \delta < 1 : \rho(T_{j_n} \cdots T_{j_1})^{1/n} < \delta\, \mathrm{JSR}(\mathcal{T})$, $T_{j_n} \cdots T_{j_1} \neq \Pi_n$,
  $n = 1, \dots, N$.

then the 🍺 *terminates, and thus, computes the JSR exactly.*

Elliptic 🍺

Theorem (Elliptic 🍺: M., Protasov)

Given a finite set of real square matrices $\mathcal{T} = \{T_1, \dots, T_J\}$. If

- *there exist finitely many s.m.p.s $\Pi_1, \dots, \Pi_N$,*
- *whose leading eigenvectors are unique and simple, up to the complex conjugate,*
- *and there exists a spectral gap at 1,*

then the Elliptic 🍺 terminates, and thus, computes the JSR exactly.

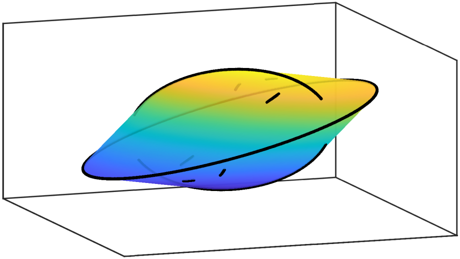
Definition of an ellipse using complex vectors

Definition

Given $a, b \in \mathbb{R}^s$, $v = a + ib$, we define

$$E(a, b) = E(v) = \{a \cos t + b \sin t : t \in [0, 2\pi]\} \subseteq \mathbb{R}^s.$$

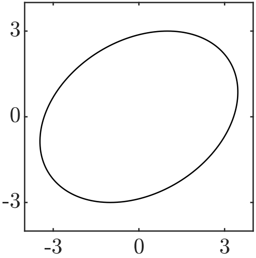
An *elliptic polytope* is the convex hull of finitely many ellipses.



UNIDENTIFIED FLYING OBJECT OBSERVED IN REGGIO CALABRIA
NOVEMBER 11^{th}, 2021.

🍺: Idea

- Given: $A = \begin{bmatrix} 1 & -4 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$
- S.m.p. candidate $\Pi = A$, $\rho := \rho(A) = \sqrt{11}$, $\mathrm{JSR}(\{A, B\}) \geq \sqrt{11}$, $\widetilde{A} = A/\rho$, $\widetilde{B} = B/\rho$.
- $v_0 = \begin{bmatrix} 1 - i\sqrt{11} \\ 3 + i0 \end{bmatrix}$, $E(\widetilde{A}v_0) = E(v_0)$.



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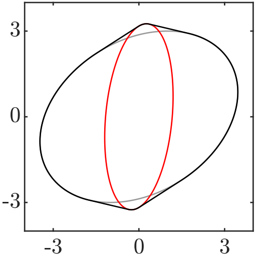

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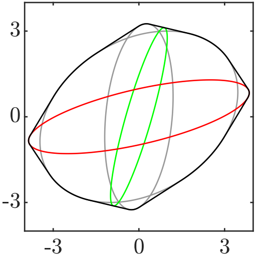

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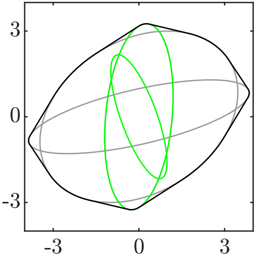

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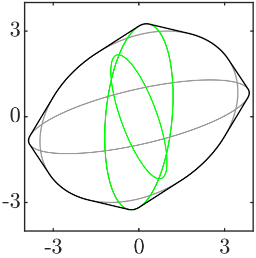


$\widetilde{A}P \subseteq P$, $\widetilde{B}P \subseteq P$,
$P = \mathrm{co}\{E(v_0), E(Bv_0), E(ABv_0)\}$.

$\Rightarrow ||\widetilde{A}||_P \leq 1$, $||\widetilde{B}||_P \leq 1$

$\Rightarrow \mathrm{JSR}(\{\widetilde{A}, \widetilde{B}\}) = \max\{||\widetilde{A}||_P, ||\widetilde{B}||_P\} \leq 1$

In particular, $||\widetilde{A}||_P = 1$.

Remaining problem:
EE - Ellipse in Ellipses

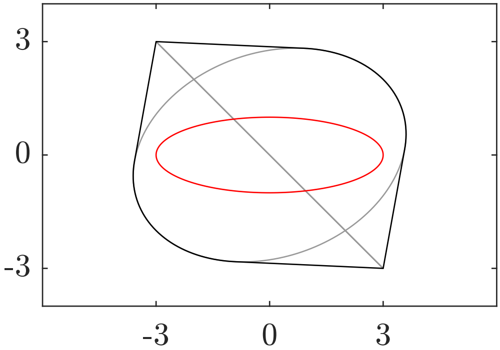
EE - Numerical computation

$$V = \left\{ \begin{bmatrix} 3-2i \\ 2+2i \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix} \right\}, p = \left\{ \begin{bmatrix} 3 \\ i \end{bmatrix} \right\}.$$

```
V   = [3-2i 2+2i; -3 3].';
pt  = [3 1i].';
nca = polytopenorm(pt,V,'cees'); % [0         0.96583]
ncb = polytopenorm(pt,V,'cee');  % [0.42434   0.84879]
ncc = polytopenorm(pt,V,'crpe'); % [0.84716   0.86386]
ncd = polytopenorm(pt,V,'cipe'); % [0.84868   0.84879]
nce = polytopenorm(pt,V,'cipp'); % [0.84874   0.84874]
```

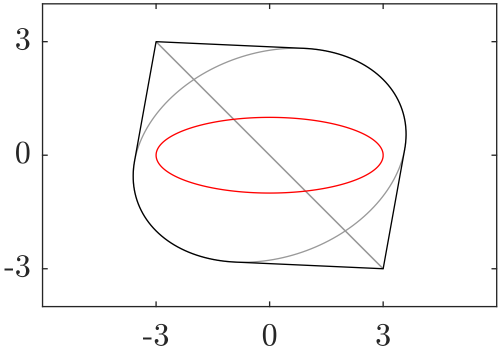
- Brown functions are part of *ttoolboxes*:
  https://gitlab.com/tommsch/ttoolboxes
- Plots done using `plotm`

EE - Ellipse in ellipses (exact)



- Conjecture: NP-hard
-

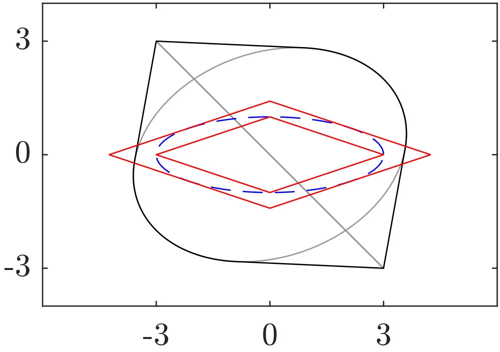
EE - Ellipse in ellipses (approximate)



- CP problem - Accuracy 10^{-5}
- Approximation factor: 0 / 0.5

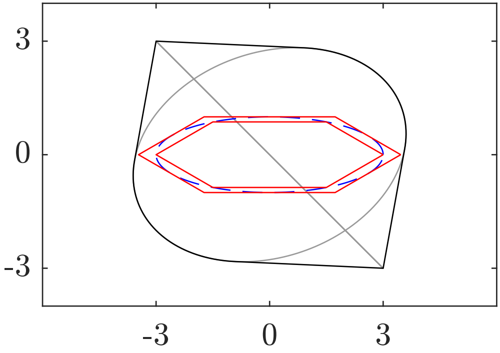
RPE - Regular Polytope in Ellipses

EE - Regular polytope in ellipses



- CP problem - Accuracy 10^{-5}
- Approximation factor: Arbitrary close to 1

EE - Regular polytope in ellipses



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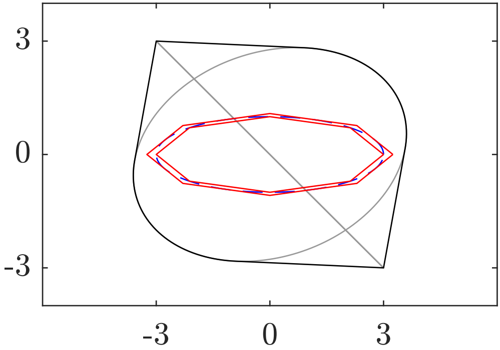

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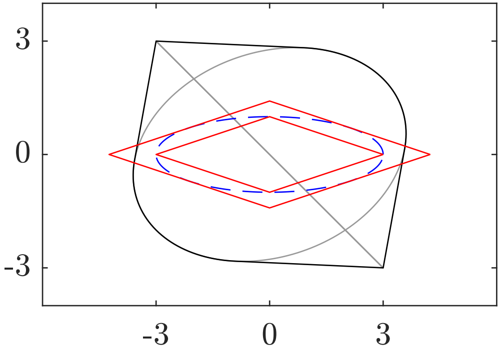
IPE - Irregular Polytope in Ellipses

EE - Irregular polytope in ellipses: M., Protasov



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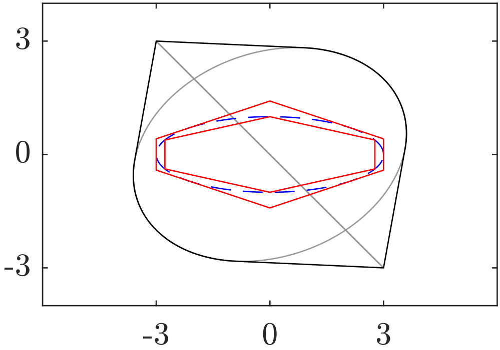
EE - Irregular polytope in ellipses: M., Protasov



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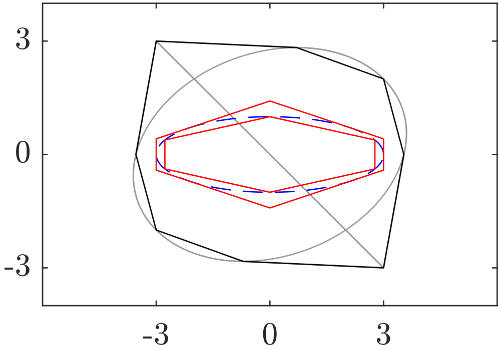
IPP - Irregular Polytope in Polytope

EE - Irregular polytope in polytope: M., Protasov



- LP problem - Accuracy 10^{-9}
- Approximation factor: Arbitrary close to 1

Open Problems

Exact computation - open problems

- S.m.p.s with multiple leading eigenvalues
- Higher dimensions
  - Polytope norm such that $\exists N : ||T_{j_N} \cdots T_{j_1}||_P < 1$.
  - Using tailored norm for tree algorithm
- Infinitely many s.m.p.s with finitely many leading eigenvectors
  - Combined numeric and symbolic computation
  - Computing JSR of a larger set of matrices: $\mathrm{JSR}(\mathcal{T}) = \mathrm{JSR}(\mathrm{co}(\mathcal{T}))$.
- ¿? Infinitely many s.m.p.s with infinitely many leading eigenvectors
- ¿? Sets of matrices without spectral gap